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Differential calculi over quantum groups and twisted cyclic cocycles

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Abstract

We study some aspects of the theory of non-commutative differential calculi over complex algebras, especially over the Hopf algebras associated to compact quantum groups in the sense of S.L. Woronowicz. Our principal emphasis is on the theory of twisted graded traces and their associated twisted cyclic cocycles. One of our principal results is a new method of constructing differential calculi, using twisted graded traces.

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1. Introduction

A compact group is a compact space with a continuous multiplication satisfying certain extra conditions. In the theory of compact quantum groups developed by S.L. Woronowicz [3,4,6,7,9], one replaces the compact space by a unital C*-algebra A that is in general non-commutative, and replaces the group multiplication by a co-multiplication on A satisfying certain cancellation conditions. Contained in A is a dense $*$-subalgebra $\mathcal{A}$, the *representation* algebra, that is a Hopf algebra under the restriction co-multiplication. Both A and $\mathcal{A}$ admit a Haar integral and this is vital for many aspects of the theory we develop in this paper.

The considerations in this paper are motivated by the theory of compact quantum groups, but it is not these objects that we study here; rather, we study differential calculi over such

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groups. Our context is therefore non-commutative differential geometry in the spirit of that subject as developed by Alain Connes [2]. The study of differential calculi in the quantum group setting was initiated by Woronowicz—indeed, he constructed the first example of such a calculus [8]. However, it was immediately apparent in his work that Connes' theory of non-commutative geometry does not cover the calculi occurring in the quantum setting. To explain briefly what is involved, recall that although the algebra of forms in the classical setting of differential manifolds is not commutative, it is "nearly" so, in the sense that $\omega\omega' = (-1)^{kl}\omega'\omega$, if ω and ω' are a k-form and an l-form, respectively. In Connes' non-commutative geometry, it is no longer true that $\omega\omega' = (-1)^{kl}\omega'\omega$. However, for a graded trace (this is an appropriate kind of "integral" on the "non-commutative manifold"), we have $\int \omega\omega' = (-1)^{kl} \int \omega'\omega$, where ω and ω' are a k-form and an l-form, respectively. This integral condition is of fundamental importance in the cyclic cocyle theory developed so successfully by Connes in the past two decades. However, even this weaker commutativity condition does not hold in the context of differential geometry over quantum groups. If one thinks of a graded trace as the analogue of a trace on a C*-algebra, then one can explain the situation in the quantum setting by saying that one must replace a trace by a KMS state. More precisely, in this setting there is an automorphism σ of degree zero of the algebra of forms such that $\int \omega\omega' = (-1)^{kl} \int \sigma(\omega')\omega$, where ω and ω' are a k-form and an l-form, respectively. This is, of course, analogous to the situation with a KMS state h on a C*-algebra, where one has an automorphism σ on a dense $*$-subalgebra for which $h(ab) = h(\sigma(b)a)$, for all elements a and b in the subalgebra.

In his seminal paper on differential calculi over quantum groups [8], Woronowicz remarks that the integral he defines on his three-dimensional calculus over the quantum group $SU_q(2)$ does not fit into the framework of Connes' non-commutative geometry, but he does not develop this observation. In this paper, we introduce the concept of a twisted graded trace (the analogue of a KMS state) to replace Connes' graded traces. It is then necessary to develop a theory of twisted cyclic cocycles and we do this here. One of our principal results is a new method of constructing differential calculi; in essence, in this approach we start with a twisted graded trace and construct a calculus (in Woronowicz's approach one goes in the opposite direction). We feel that our approach may be more natural, since, to some extent, it involves giving a "presentation" of the calculus in terms of generators and relations.

We give a brief overview of the paper now. In Section 2 we introduce the basic terminology and prove two theorems that are very useful for constructing twisted graded traces. We also introduce a quotient construction for obtaining a differential calculus from a twisted graded trace. In Section 3 we introduce twisted cyclic cocycles and develop their relationship with twisted graded traces. In both this section and the next, we develop a theory of twisted cyclic cohomology. This contains Connes' theory as a special case, but, as we have indicated above, the more general theory is necessary to deal with the examples that occur in the quantum group setting. However, the theory developed in Sections 2–4 is not restricted to the quantum group setting and applies in the more general context of differential calculi over arbitrary unital algebras. In Section 5 we develop aspects of the theory of left-invariant twisted graded traces over left-covariant differential calculi. In this situation the underlying algebra is assumed to be a Hopf algebra. An important result here is that the differential calculus constructed from a left-invariant twisted graded trace on the universal calculus is shown to be itself left-covariant. Also, we give a characterization of the twisted cyclic cocycles that correspond to left-invariant twisted graded traces. In the final section, Section 6, we show how our ideas can be used to give an alternative construction of Woronowicz's first, three-dimensional, differential calculus over quantum $\mathrm{SU}(2)$.

2. Differential calculi

In this section, we set up the basic terminology for studying differential calculi over algebras that are not necessarily commutative. One can think of this as the study of differential forms in the setting of quantum spaces or manifolds. We give a general procedure for constructing such calculi. We begin by recalling some basic definitions.

Let Ω be a (positively) graded algebra, $\Omega = \oplus_{n=0}^{\infty}\Omega_n$. A *graded derivation* on Ω is a linear map $d : \Omega \to \Omega$ for which $d(\omega'\omega) = d(\omega')\omega + (-1)^n\omega'\,\mathrm{d}\omega$, for all $\omega' \in \Omega_n$ and all $\omega \in \Omega$.

A *graded differential algebra* is a pair (Ω, d), where Ω is a graded algebra, d is a graded derivation on Ω of degree 1 (as a linear map) and $d^2 = 0$. The elements of Ω are referred to as the *forms* of (Ω, d) and the elements of Ω_n as the *n-forms*. The operator d is referred to as the *differential*.

Now suppose that $\mathcal{A}$ is an arbitrary associative unital algebra. Then there is a graded differential algebra $(\bar{\Omega}, d)$, for which $\bar{\Omega}_0 = \mathcal{A}$, that has the following universal property: If σ is an algebra homomorphism from $\mathcal{A}$ into the algebra Ω_0 of 0-forms of a graded differential algebra (Ω, d), then there exists a unique algebra homomorphism $\bar{\sigma}$ from $\bar{\Omega}$ to Ω extending σ such that $\bar{\sigma}d = d\bar{\sigma}$. This property uniquely determines $(\bar{\Omega}, d)$ (up to isomorphism). Note that $\bar{\sigma}$ is clearly necessarily of grade zero. We shall usually denote the extension $\bar{\sigma}$ by the same symbol σ as the original homomorphism.

We shall use the following two useful properties of $(\bar{\Omega}, d)$:

(1) Let $n \geq 1$. Then every element of $\bar{\Omega}_n$ is a sum of elements of the form $a_0\,\mathrm{d}a_1\cdots\mathrm{d}a_n$, and $\mathrm{d}a_1\cdots\mathrm{d}a_n$, where the elements $a_0, a_1, \ldots, a_n$ belong to $\mathcal{A}$;
(2) Let n be a positive integer and T_1 a multilinear map from $\mathcal{A}^{n+1}$ to a linear space Y and T_2 a linear map from $\mathcal{A}^n$ to the same linear space Y. Then there is a unique linear map $\hat{T}$ from $\bar{\Omega}_n$ to Y for which $\hat{T}(a_0\,\mathrm{d}a_1\cdots\mathrm{d}a_n) = T_1(a_0, a_1, \ldots, a_n)$ and $\hat{T}(\mathrm{d}a_1\cdots\mathrm{d}a_n) = T_2(a_1, \ldots, a_n)$, for all $a_0, a_1, \ldots, a_n \in \mathcal{A}$.

In practice, the universal graded differential algebra $(\bar{\Omega}, d)$ is too big to be useful. However, it can be used to construct smaller, finite-dimensional differential algebras that are useful.

A *differential calculus* over $\mathcal{A}$ is a graded differential algebra (Ω, d) for which

(1) $\Omega_0 = \mathcal{A}$;
(2) Let $n \geq 1$. Then every element of Ω_n is a sum of elements of the form $a_0\,\mathrm{d}a_1\cdots\mathrm{d}a_n$ and $\mathrm{d}a_1\cdots\mathrm{d}a_n$, where the elements $a_0, a_1, \ldots, a_n$ belong to $\mathcal{A}$.

If the differential calculus Ω is unital (as an algebra), then the unit of Ω has to belong to $\Omega_0 = \mathcal{A}$ and therefore has to be equal to the unit 1 of $\mathcal{A}$.

We shall say the differential calculus (Ω, d) is *finite-dimensional*, of *dimension* N, if $\Omega_N \neq 0$ and $\Omega_n = 0$ for $n > N$.

The universal graded differential algebra is clearly a differential calculus over $\mathcal{A}$, but it is, equally clearly, not finite-dimensional, nor unital.

We now describe a general procedure for obtaining a new, "smaller" calculus from a given calculus. Let N be a positive integer and let (Ω, d) be a differential calculus over $\mathcal{A}$ that is either not finite-dimensional, or is of finite dimension greater than N. We define a new differential calculus (Ω', d') of dimension N by setting $\Omega'_k = \Omega_k$, if $k \leq N$ and $\Omega'_k = 0$, if $k > N$. We define the multiplication $\cdot$ in Ω' by setting, for $\omega_1 \in \Omega_k$ and $\omega_2 \in \Omega_l$, $\omega_1 \cdot \omega_2 = \omega_1\omega_2$, if $k + l \leq N$, and by setting $\omega_1 \cdot \omega_2 = 0$ if $k + l > N$. We set $d'(\omega_1) = d(\omega_1)$, if $k \leq N$ and set $d'(\omega_1) = 0$, if $k > N$. We call (Ω', d') the differential calculus of dimension N *obtained from* (Ω, d) *by truncation*.

If (Ω, d) is a differential calculus over $\mathcal{A}$, we say a linear functional $\int$ on Ω is *closed* if $\int d = 0$. If $\omega_1, \ldots, \omega_M \in \Omega$, then a simple induction shows that $\mathrm{d}\omega_1\, \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_M = d(\omega_1\, \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_M)$. Hence, if $\int$ is closed, $\int \mathrm{d}\omega_1\, \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_M = 0$. We shall frequently tacitly make use of this observation. If ω is a k-form and ω' an arbitrary form, then $\int (\mathrm{d}\omega)\omega' = (-1)^{k+1} \int \omega\, \mathrm{d}\omega'$, another result we shall use tacitly in the sequel. It follows from the fact that $d(\omega\omega') = (\mathrm{d}\omega)\omega' + (-1)^k \omega\, \mathrm{d}\omega'$ and $\int d(\omega\omega') = 0$.

A linear functional $\int$ on Ω is a *twisted graded trace* if there is an algebra automorphism $\sigma : \Omega \to \Omega$ of degree zero for which $\sigma d = d\sigma$ and $\int \omega'\omega = (-1)^{kl} \int \sigma(\omega)\omega'$, for all non-negative integers k and l and for all $\omega \in \Omega_k$ and $\omega' \in \Omega_l$.

We say σ is a *twist automorphism* associated to $\int$. It is useful to observe that $\int \sigma(\omega) = \int \omega$, for all $\omega \in \Omega$. To see this, observe first that $a = a1$ and $\mathrm{d}a = d(a1) = (\mathrm{d}a)1 + a(d1)$ for all $a \in A$. It follows that any element of Ω is a sum of products of two elements of Ω. Let $\omega, \omega' \in \Omega$. We may write $\omega = \sum_k \omega_k$ and $\omega' = \sum_k \omega'_k$, where $\omega_k, \omega'_k \in \Omega_k$. Then $\int \omega\omega' = \sum_{k,l} \int \omega_k\omega'_l = \sum_{k,l} (-1)^{kl} \int \sigma(\omega'_l)\omega_k = \sum_{k,l} \int \sigma(\omega_k)\sigma(\omega'_l) = \int \sigma(\omega)\sigma(\omega') = \int \sigma(\omega\omega')$.

Theorem 2.1. *Let $(\bar{\Omega}, d)$ be the universal calculus over a unital algebra $\mathcal{A}$. Suppose that $\int$ is a closed linear functional on $\bar{\Omega}$ and that $\sigma_0 : \mathcal{A} \to \mathcal{A}$ is an algebra automorphism for which $\int \sigma_0(a)\omega = \int \omega a$, for all $a \in \mathcal{A}$ and $\omega \in \bar{\Omega}$. Then $\int$ is a twisted graded trace having a twist automorphism σ that extends σ_0.*

Proof. The automorphism, $\sigma_0 : \bar{\Omega}_0 \to \bar{\Omega}_0$, extends uniquely to an automorphism, $\sigma : \bar{\Omega} \to \bar{\Omega}$, for which $\sigma d = d\sigma$, by the universal property of $(\bar{\Omega}, d)$. We shall show that $\int$ is a twisted graded trace, with σ as its twist automorphism. Thus, to prove the theorem, we have only to show that, for each positive integer N,

$$\int \omega'\omega = (-1)^{k(N-k)} \int \sigma(\omega)\omega', \tag{2.1}$$

for all integers k such that $0 \leq k \leq N$, and for all $\omega \in \bar{\Omega}_k$ and $\omega' \in \bar{\Omega}_{N-k}$. We shall prove this by induction on k. It clearly holds for $k = 0$ by hypothesis. Let's assume it holds for k and we shall prove it for $k + 1$, where we also suppose that $k + 1 \leq N$. We first show that

$$\int \alpha\, \mathrm{d}\omega = (-1)^{(k+1)(N-k-1)} \int \sigma(\mathrm{d}\omega)\alpha, \tag{2.2}$$

where $\omega \in \bar{\Omega}_k$ and $\alpha \in \bar{\Omega}_{N-k-1}$. We suppose first that $k + 1 < N$. If $\alpha = \mathrm{d}\omega'$, where $\omega' \in \bar{\Omega}_{N-k-2}$, the closeness of $\int$ implies that both sides of the above equation are 0 and hence equal. Since $\bar{\Omega}_{N-k-1}$ is the linear span of elements of the form $\mathrm{d}\omega'$ and $(\mathrm{d}\omega')a$, where $\omega' \in \bar{\Omega}_{N-k-2}$ and $a \in \mathcal{A}$, we may now clearly suppose that $\alpha = (\mathrm{d}\omega')a$. We have $\int (\mathrm{d}\omega')a\,\mathrm{d}\omega = \int \mathrm{d}\omega'\,\mathrm{d}(a\omega) - \int (\mathrm{d}\omega')(\mathrm{d}a)\omega = -\int (\mathrm{d}\omega')(\mathrm{d}a)\omega = (-1)^{1+k(N-k)} \int \sigma(\omega)(\mathrm{d}\omega')\,\mathrm{d}a$, by the inductive hypothesis. Since $d(\sigma(\omega)\omega') = (d\sigma(\omega))\omega' + (-1)^k\sigma(\omega)\,\mathrm{d}\omega' = \sigma(\mathrm{d}\omega)\omega' + (-1)^k\sigma(\omega)\,\mathrm{d}\omega'$, we get

$$\begin{aligned}
\int (\mathrm{d}\omega')a\,\mathrm{d}\omega &= (-1)^{1+k(N-k)}(-1)^k\left[\int d(\sigma(\omega)\omega')\,\mathrm{d}a - \int \sigma(\mathrm{d}\omega)\omega'\,\mathrm{d}a\right] \\
&= (-1)^{1+k(N-k)}(-1)^{k+1}\int \sigma(\mathrm{d}\omega)\omega'\,\mathrm{d}a \\
&= (-1)^{1+k(N-k)}(-1)^{k+1}(-1)^{N-k-2} \\
&\quad\times\left[\int \sigma(\mathrm{d}\omega)d(\omega'a) - \int \sigma(\mathrm{d}\omega)(\mathrm{d}\omega')a\right] \\
&= (-1)^{1+k(N-k)}(-1)^{k+1}(-1)^{N-k-1}\int \sigma(\mathrm{d}\omega)(\mathrm{d}\omega')a \\
&= (-1)^{(k+1)(N-k-1)}\int \sigma(\mathrm{d}\omega)(\mathrm{d}\omega')a.
\end{aligned}$$

This shows that Eq. (2.2) holds, as required, when $k+1 < N$. For $k+1 = N$ the argument is similar, but much simpler, and is therefore omitted. It follows now from Eq. (2.2) that, for all $a \in \mathcal{A}$, we have

$$\begin{aligned}
\int \alpha a\,\mathrm{d}\omega &= (-1)^{(k+1)(N-k-1)}\int \sigma(\mathrm{d}\omega)\alpha a \\
&= (-1)^{(k+1)(N-k-1)}\int \sigma_0(a)\sigma(\mathrm{d}\omega)\alpha = \int \sigma(a\,\mathrm{d}\omega)\alpha.
\end{aligned}$$

This shows that Eq. (2.1) is satisfied for k in place of $k+1$. This completes our induction, so Eq. (2.1) is now seen to be true for $k = 0, \ldots, N$. □

We say that a linear functional $\int$ on Ω is *left faithful* if, whenever $\omega \in \Omega$ is such that $\int \omega'\omega = 0$, for all $\omega' \in \Omega$, we necessarily have $\omega = 0$.

Theorem 2.2. *Suppose (Ω, d) is a differential calculus over a unital algebra $\mathcal{A}$. Suppose that $\int$ is a left faithful, closed linear functional on Ω and that $\sigma_0 : \mathcal{A} \to \mathcal{A}$ is an algebra automorphism for which $\int \sigma_0(a)\omega = \int \omega a$, for all $a \in \mathcal{A}$ and $\omega \in \Omega$. Then $\int$ is a twisted graded trace having a twist automorphism σ that extends σ_0.*

Proof. The automorphism, $\sigma_0 : \bar{\Omega}_0 \to \bar{\Omega}_0$, extends uniquely to an automorphism, $\bar{\sigma} : \bar{\Omega} \to \bar{\Omega}$, for which $\bar{\sigma}d = d\bar{\sigma}$, by the universal property of the universal differential calculus $(\bar{\Omega}, d)$. Likewise the isomorphism, $\mathrm{id}_{\mathcal{A}} : \bar{\Omega}_0 \to \Omega_0$, extends uniquely to a surjective homomorphism, $\pi : \bar{\Omega} \to \Omega$, such that $\pi d = d\pi$. We define $\int'$ on $\bar{\Omega}$ by setting $\int' \omega = \int \pi(\omega)$, for all $\omega \in \bar{\Omega}$. Clearly, $\int'$ is a closed, linear functional on $\bar{\Omega}$ satisfying the hypothesis of the preceding theorem. Hence, $\int'$ is a twisted graded trace, with $\bar{\sigma}$ as its twist automorphism.

Suppose now that $\omega \in \bar{\Omega}$ and $\pi(\omega) = 0$. We shall show that $\pi(\bar{\sigma}(\omega)) = 0$. If $\omega' \in \bar{\Omega}$, then $\int \pi(\bar{\sigma}(\omega'))\pi(\bar{\sigma}(\omega)) = \int' \bar{\sigma}(\omega'\omega) = \int' \omega'\omega = \int \pi(\omega')\pi(\omega) = 0$, since $\pi(\omega) = 0$. It follows from faithfulness of $\int$ that $\pi(\bar{\sigma}(\omega)) = 0$, as required.

We can now use this invariance of $\ker(\pi)$ under $\bar{\sigma}$ to induce a homomorphism σ on Ω defined by setting $\sigma(\pi(\omega)) = \pi(\bar{\sigma}(\omega))$, for all $\omega \in \bar{\Omega}$. It is clear that $\int \omega'\omega = (-1)^{kl} \int \sigma(\omega)\omega'$, for all integers k and l and for all $\omega \in \Omega_k$ and $\omega' \in \Omega_l$. Clearly, since $\bar{\sigma}$ extends σ_0, so does σ. It is easily checked that $\sigma d = d\sigma$. Moreover, σ is surjective, since $\bar{\sigma}$ and π are. Thus, to show that $\int$ is a twisted graded trace with σ as twist automorphism, we need only show now that σ is injective. To see this, suppose that $\omega \in \Omega_k$ and $\sigma(\omega) = 0$. Then $\int \omega'\omega = (-1)^{kl} \int \sigma(\omega)\omega' = 0$, for all integers l and for all $\omega' \in \Omega_l$. Hence, since $\int$ is left faithful, $\omega = 0$. Therefore, σ is injective, as required. □

If $\int$ is a linear functional on a differential calculus, its *left kernel* is defined to be the set of all forms ω for which $\int \omega'\omega = 0$, for all $\omega' \in \Omega$. Obviously, the left kernel is a left ideal of Ω. If the intersection of the left kernel of $\int$ with $\mathcal{A}$ is the zero space, we say $\int$ is *weakly faithful*. Obviously, $\int$ is left faithful if, and only if, its left kernel is the zero space; hence, $\int$ is weakly faithful if it is left faithful, as one would expect.

Theorem 2.3. *Let $\int$ be a twisted graded trace on a differential calculus (Ω, d) over a unital algebra $\mathcal{A}$.*

(1) *$\int$ is weakly faithful if, and only if, for each element $a \in \mathcal{A}$ for which $\int a\omega = 0$, for all $\omega \in \Omega$, we have $a = 0$.*
(2) *If $\int$ is weakly faithful, then $\int$ admits exactly one twist automorphism.*

Proof. First, suppose that $\int$ is weakly faithful. Let σ be any twist automorphism of $\int$ and suppose that $a \in \mathcal{A}$ and that $\int a\omega = 0$, for all $\omega \in \Omega$. Then $\int \omega\sigma^{-1}(a) = 0$. Hence, by weak faithfulness of $\int$, $\sigma^{-1}(a) = 0$ and therefore, $a = 0$. This shows the forward implication in Condition (1) and the reverse implication is shown by similar reasoning.

To see Condition (2) holds, let ρ and σ be twist automorphisms for $\int$. Then, for all $a \in \mathcal{A}$ and $\omega \in \Omega$, $\int(\rho(a) - \sigma(a))\omega = \int \rho(a)\omega - \int \sigma(a)\omega = \int \omega a - \int \omega a = 0$. Hence, $\rho(a) = \sigma(a)$. Using the fact that $\rho d = d\rho$ and $\sigma d = d\sigma$, it now follows immediately that $\rho = \sigma$. □

Let N be a non-negative integer. We say that a linear functional $\int$ on Ω is *N-dimensional* if $\int \omega = 0$, for all k-forms, where $k \neq N$.

Suppose now $\int'$ is an N-dimensional, weakly faithful, closed twisted graded trace on a differential calculus $(\hat{\Omega}, d)$ over $\mathcal{A}$ and let $\hat{\sigma}$ denote the twist automorphism of $\int'$. We are going to construct a new, N-dimensional, differential calculus (Ω, d) from $(\hat{\Omega}, d, \int')$ and a new, N-dimensional, closed twisted graded trace $\int$ on Ω that is left faithful.

The twisted tracial property of $\int'$ implies that, for each form $\omega \in \hat{\Omega}$, the condition $\int \omega'\omega = 0$, for all $\omega' \in \hat{\Omega}$, is equivalent to the condition $\int \omega\omega' = 0$, for all $\omega' \in \hat{\Omega}$. Hence, if I is the left kernel of $\int'$, it is not only a left ideal of $\hat{\Omega}$, but is also a right ideal. We denote by Ω the quotient algebra $\hat{\Omega}/I$. It is trivially verified that $\hat{\Omega}_n \subseteq I$ for all $n > N$ and that if $\omega \in I$, then its kth component ω_k belongs to I also. It follows that if Ω_k denotes the image of $\hat{\Omega}_k$ in the quotient algebra Ω, then $\Omega = \Omega_0 \oplus \cdots \oplus \Omega_N$. Moreover, this makes Ω into a graded algebra. Since $I \cap \mathcal{A} = 0$, because $\int'$ is weakly faithful, we may, and we do, identify Ω_0 with $\mathcal{A}$.

If $\omega' \in \hat{\Omega}_k$ and $\omega \in I$, then $\int' \omega' \,\mathrm{d}\omega = (-1)^{k+1} \int' (\mathrm{d}\omega')\omega = 0$. This implies that $\mathrm{d}\omega \in I$. Hence, $d(I) \subseteq I$ and therefore d induces a linear map $d : \Omega \to \Omega$. It is immediate that d is a graded derivation on Ω and, indeed, that (Ω, d) is an N-dimensional differential calculus over $\mathcal{A}$.

Since $\int'$ clearly annihilates I, we get an induced linear map $\int$ on Ω. Also, it is clear that $\hat{\sigma}(I) \subseteq I$, so that $\hat{\sigma}$ induces an algebra automorphism σ on Ω. It is now easily verified that $\int$ is an N-dimensional, closed twisted graded trace on Ω with σ as its twist automorphism.

We call (Ω, d) the *differential calculus* associated to $(\hat{\Omega}, d, \int')$ and $\int$ the *canonical* twisted graded trace on Ω. The significant gains resulting from this construction are that (Ω, d) is finite-dimensional and that $\int$ is left faithful.

It is straightforward to verify that if one starts with an N-dimensional differential calculus (Ω, d) over $\mathcal{A}$, and with a left faithful, closed twisted graded trace $\int$ on Ω, then (up to isomorphism) one can obtain Ω, d and $\int$ by the preceding quotient construction from an N-dimensional, weakly faithful, closed twisted graded trace $\int'$ on $(\bar{\Omega}, d)$.

The question now arises as to how we can obtain twisted graded traces on $(\bar{\Omega}, d)$. We shall see these arise from twisted cyclic cocycles. We shall discuss these objects and explain their relationship with twisted graded traces in Section 3.

Suppose now that $\mathcal{A}$ is a unital $*$-algebra. We shall say that (Ω, d) is a $*$-*differential calculus* over $\mathcal{A}$ if it is a differential calculus over $\mathcal{A}$ and if Ω is endowed with a conjugate-linear map, $\Omega \to \Omega$, $\omega \mapsto \omega^*$, extending the involution on $\mathcal{A}$, having the following properties:

(1) $(\omega^*)^* = \omega$, for all $\omega \in \Omega$;
(2) $(\omega_1\omega_2)^* = (-1)^{kl}\omega_2^*\omega_1^*$, for all $\omega_1 \in \Omega_k$ and $\omega_2 \in \Omega_l$;
(3) $d(\omega^*) = (\mathrm{d}\omega)^*$, for all $\omega \in \Omega$.

We shall call the map, $\omega \mapsto \omega^*$, the *graded involution* of Ω. Notice that there is at most one such graded involution.

A linear map, $\int : \Omega \to \mathbf{C}$, is *self-adjoint* if $\int \omega^* = (\int \omega)^-$, for all $\omega \in \Omega$.

The universal differential calculus $(\bar{\Omega}, d)$ of a $*$-algebra $\mathcal{A}$ is a $*$-differential calculus in a natural way. Suppose now $\int'$ is an N-dimensional, weakly faithful, self-adjoint, closed twisted graded trace on $(\bar{\Omega}, d)$. Let I be its left kernel, (Ω, d) the associated N-dimensional differential calculus and $\int$ the canonical twisted graded trace on Ω. Then I is self-adjoint—that is, if $\omega \in I$, then $\omega^* \in I$—and (Ω, d) is a $*$-differential calculus over $\mathcal{A}$, where $(\omega + I)^* = \omega^* + I$, for all $\omega \in \bar{\Omega}$. To see I is self-adjoint, suppose that ω is a k-form belonging to I. If ω' is an $(N-k)$-form, then $\int' \omega'\omega^* = (-1)^{k(N-k)}(\int' \omega(\omega')^*)^- = 0$. Hence, $\omega^* \in I$. This proves $I^* \subseteq I$. It now follows easily that the involution $(\omega + I)^* = \omega^* + I$ makes (Ω, d) into a $*$-differential calculus over $\mathcal{A}$. It is equally easy to see that $\int$ is self-adjoint.

3. Twisted cyclic cocycles and differential calculi

Suppose that $\mathcal{A}$ is a unital algebra. For $n \geq 0$, let $\mathbf{C}^n(\mathcal{A})$ denote the set of all multilinear maps from $\mathcal{A}^{n+1}$ to $\mathbf{C}$. Set $\mathbf{C}^*(\mathcal{A}) = \oplus_{n\in\mathbf{N}}\mathbf{C}^n(\mathcal{A})$. Then $\mathbf{C}^*(\mathcal{A})$ is a graded linear space. There exists a unique linear map, $\mathbf{b} : \mathbf{C}^*(\mathcal{A}) \to \mathbf{C}^*(\mathcal{A})$, making $(\mathbf{C}^*(\mathcal{A}), \mathbf{b})$ a cochain complex for which, for $\varphi \in \mathbf{C}^n(\mathcal{A})$,

$$\begin{aligned}(\mathbf{b}\varphi)(a_0,\dots,a_{n+1}) &= \sum_{i=0}^{n}(-1)^i\varphi(a_0,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_{n+1})\\ &\quad+(-1)^{n+1}\varphi(a_{n+1}a_0,a_1,\dots,a_n).\end{aligned}$$

The *Hochschild cohomology* $\mathbf{HH}^*(\mathcal{A})$ of $\mathcal{A}$ is defined to be the cohomology of $(\mathbf{C}^*(\mathcal{A}),\mathbf{b})$. Thus, $\mathbf{HH}^n(\mathcal{A}) = \mathbf{H}^n(\mathbf{C}^*(\mathcal{A}),\mathbf{b})$ for all $n \in \mathbf{Z}$ (where we understand it to be zero if n is negative).

The *permutation operator* λ on $\mathbf{C}^*(\mathcal{A})$ is the linear isomorphism of degree zero, defined by setting $\lambda(\varphi)(a_0,a_1,\dots,a_n) = (-1)^n\varphi(a_n,a_0,a_1,\dots,a_{n-1})$, for $n \geq 0$, $\varphi \in \mathbf{C}^n(\mathcal{A})$ and $a_0,\dots,a_n \in \mathcal{A}$. Set $\mathbf{C}^*_\lambda(\mathcal{A}) = \oplus_{n\in\mathbf{N}}\mathbf{C}^n_\lambda(\mathcal{A})$, where $\mathbf{C}^n_\lambda(\mathcal{A}) = \{\varphi \in \mathbf{C}^n(\mathcal{A})|\lambda(\varphi) = \varphi\}$. The coboundary operator $\mathbf{b}$ leaves each space $\mathbf{C}^n_\lambda(\mathcal{A})$ invariant and therefore its restriction makes $(\mathbf{C}^*_\lambda(\mathcal{A}),\mathbf{b})$ into a cochain complex. The cohomology of this complex is denoted by $\mathbf{H}^*_\lambda(\mathcal{A})$ and called the *cyclic cohomology* of $\mathcal{A}$. Thus, $\mathbf{H}^n_\lambda(\mathcal{A}) = \mathbf{H}^n(\mathbf{C}^*_\lambda(\mathcal{A}),\mathbf{b})$.

It will be useful to recall also the degree 1 operator $\mathbf{b}'$ on $\mathbf{C}^*(\mathcal{A})$ defined by the formula

$$(\mathbf{b}'\varphi)(a_0,\dots,a_{n+1}) = \sum_{i=0}^{n}(-1)^i\varphi(a_0,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_{n+1})$$

for $n \geq 0$ and $\varphi \in \mathbf{C}^n(\mathcal{A})$. It is well-known that $(\mathbf{b}')^2 = 0$ and that the cohomology of the cochain complex $(\mathbf{C}^*(\mathcal{A}),\mathbf{b}')$ is trivial, $\mathbf{H}^*(\mathbf{C}^*(\mathcal{A}),\mathbf{b}') = 0$.

We generalize the definition of cyclic cohomology now. Suppose that $(\mathcal{A},\sigma)$ is a pair consisting of a unital algebra $\mathcal{A}$ and an algebra automorphism $\sigma : \mathcal{A} \to \mathcal{A}$. We get a new operator corresponding to the permutation operator, a linear isomorphism $\lambda : \mathbf{C}^*(\mathcal{A}) \to \mathbf{C}^*(\mathcal{A})$ of degree zero, by setting

$$\lambda(\varphi)(a_0,a_1,\dots,a_n) = (-1)^n\varphi(\sigma(a_n),a_0,a_1,\dots,a_{n-1})$$

for $n \geq 0$ and $\varphi \in \mathbf{C}^n(\mathcal{A})$. We set $\mathbf{C}^*_\lambda(\mathcal{A},\sigma) = \oplus_{n\in\mathbf{N}}\mathbf{C}^n_\lambda(\mathcal{A},\sigma)$, where $\mathbf{C}^n_\lambda(\mathcal{A},\sigma) = \{\varphi \in \mathbf{C}^n(\mathcal{A})|\lambda(\varphi) = \varphi\}$. We shall make $\mathbf{C}^*_\lambda(\mathcal{A},\sigma)$ into a cochain complex whose cohomology will be a "twisted" version of ordinary cyclic cohomology. To this end we introduce new operators $\mathbf{c}$ and $\mathbf{b}$ on $\mathbf{C}^*(\mathcal{A})$, both of degree 1. These are defined by setting $\mathbf{b} = \mathbf{b}' + \mathbf{c}$, where, for $\varphi \in \mathbf{C}^n(\mathcal{A})$, and $a_0,\dots,a_n \in \mathcal{A}$,

$$(\mathbf{c}\varphi)(a_0,\dots,a_{n+1}) = (-1)^{n+1}\varphi(\sigma(a_{n+1})a_0,a_1,\dots,a_n).$$

Thus, $\mathbf{b}$ is a "twisted" version of the usual Hochschild coboundary operator. To see that $\mathbf{b}^2 = 0$, one uses the fact that $(\mathbf{b}')^2 = 0$ and proves the easily verified fact that $\mathbf{c}\mathbf{b}' + \mathbf{b}'\mathbf{c} + \mathbf{c}^2 = 0$. As in the classical cyclic cocycle theory, one can show that $\mathbf{b}'(1-\lambda) = (1-\lambda)\mathbf{b}$. This immediately implies that $\mathbf{C}^*_\lambda(\mathcal{A},\sigma) = \{\varphi \in \mathbf{C}^*(\mathcal{A})|\lambda\varphi = \varphi\}$ is invariant under $\mathbf{b}$. Hence, by restricting $\mathbf{b}$, we get a cochain complex $(\mathbf{C}^*_\lambda(\mathcal{A},\sigma),\mathbf{b})$. We denote by $\mathbf{H}^*_\lambda(\mathcal{A},\sigma)$ the cohomology of this complex and call it the *twisted cyclic cohomology* of $(\mathcal{A},\sigma)$. We denote by $\mathbf{Z}^n_\lambda(\mathcal{A},\sigma)$ and $\mathbf{B}^n_\lambda(\mathcal{A},\sigma)$ the n-cocyles and n-coboundaries for the complex $(\mathbf{C}^*_\lambda(\mathcal{A},\sigma),\mathbf{b})$. We call the elements of these spaces the *twisted cyclic n-cocyles* and *n-coboundaries* of $(\mathcal{A},\sigma)$, respectively.

Clearly, if $\sigma = \mathrm{id}_{\mathcal{A}}$, then $\mathbf{H}^*_\lambda(\mathcal{A}, \sigma) = \mathbf{H}^*_\lambda(\mathcal{A})$.

Theorem 3.1. *Let (Ω, d) be a differential calculus over a unital algebra $\mathcal{A}$ and suppose that $\int$ is an N-dimensional, closed, twisted graded trace on Ω. Define the function, $\varphi : \mathcal{A}^{N+1} \to \mathbf{C}$, by setting*

$$\varphi(a_0, \dots, a_N) = \int a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_N.$$

Let σ be an automorphism of $\mathcal{A}$ for which $\int \sigma(a)\omega = \int \omega a$, for all $a \in \mathcal{A}$ and $\omega \in \Omega_N$. Then φ belongs to $\mathbf{Z}^N_\lambda(\mathcal{A}, \sigma)$.

Proof. We show first that $\lambda\varphi = \varphi$. Let $a_0, \dots, a_N$ be elements of $\mathcal{A}$. Then, since $\int$ is closed, and $\mathrm{d}a_0 \cdots \mathrm{d}a_{N-1} = d(a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{N-1})$, we have

$$\begin{aligned}\lambda\varphi(a_0, \dots, a_N) &= (-1)^N \int \sigma(a_N)\, \mathrm{d}a_0 \cdots \mathrm{d}a_{N-1} = (-1)^N \int (\mathrm{d}a_0 \cdots \mathrm{d}a_{N-1}) a_N \\ &= \int a_0 (\mathrm{d}a_1 \cdots \mathrm{d}a_{N-1})\, \mathrm{d}a_N = \varphi(a_0, \dots, a_N).\end{aligned}$$

To show that $\mathbf{b}\varphi = 0$, we shall use the fact that

$$\begin{aligned}&\sum_{i=1}^{N} (-1)^i\, \mathrm{d}a_1 \cdots \mathrm{d}(a_i a_{i+1}) \cdots \mathrm{d}a_{N+1} \\ &\quad = (-1)^N (\mathrm{d}a_1 \cdots \mathrm{d}a_N) a_{N+1} - a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_{N+1},\end{aligned} \tag{3.1}$$

for all $a_1, \dots, a_{N+1} \in \mathcal{A}$ (this is well-known, see [2, p. 187]). It follows from this equality, and from the twisted tracial property of $\int$, that

$$\begin{aligned}\mathbf{b}\varphi(a_0, \dots, a_{N+1}) &= \sum_{i=1}^{N} (-1)^i \int a_0\, \mathrm{d}a_1 \cdots \mathrm{d}(a_i a_{i+1}) \cdots \mathrm{d}a_{N+1} \\ &\quad + \int a_0 a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_{N+1} + (-1)^{N+1} \int \sigma(a_{N+1}) a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_N \\ &= \int a_0((-1)^N (\mathrm{d}a_1 \cdots \mathrm{d}a_N) a_{N+1} - a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_{N+1}) \\ &\quad + \int a_0 a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_{N+1} + (-1)^{N+1} \int a_0 (\mathrm{d}a_1 \cdots \mathrm{d}a_N) a_{N+1} = 0.\end{aligned}$$

The theorem is now proved. □

We call φ the twisted cyclic cocycle *associated to* (Ω, d) and $\int$.

Theorem 3.2. *Let σ be an automorphism of a unital algebra $\mathcal{A}$ and let $\varphi \in \mathbf{Z}^N_\lambda(\mathcal{A}, \sigma)$, for some integer $N \geq 0$. Then there exists an N-dimensional differential calculus (Ω, d) over $\mathcal{A}$ and an N-dimensional, closed twisted graded trace $\int$ on Ω such that φ is the twisted cyclic cocycle associated to (Ω, d) and $\int$.*

Proof. Define an N-dimensional linear functional $\int'$ on the universal differential calculus $\bar{\Omega}$ over $\mathcal{A}$ by setting $\int' a_0\,\mathrm{d}a_1\cdots \mathrm{d}a_N = \varphi(a_0,\dots,a_N)$ and $\int' \mathrm{d}a_1\cdots\mathrm{d}a_N = 0$, for all $a_0,\dots,a_N\in\mathcal{A}$. By definition, $\int'$ is closed.

Next we show that $\int' \omega a_{N+1} = \int' \sigma(a_{N+1})\omega$, for all $a_{N+1}\in\mathcal{A}$ and all $\omega\in\bar{\Omega}$. Clearly, to show this, we may suppose that $\omega = a_0\mathrm{d}a_1\cdots\mathrm{d}a_N$ or $\omega = \mathrm{d}a_1\cdots\mathrm{d}a_N$, for some elements $a_0,\dots,a_N\in\mathcal{A}$. Then, using the fact that $\mathbf{b}\varphi = 0$ and therefore, $\mathbf{b}'\varphi = -\mathbf{c}\varphi$, and again using Eq. (3.1), we have

$$\begin{aligned}
&\int' \sigma(a_{N+1})a_0\,\mathrm{d}a_1\cdots\mathrm{d}a_N\\
&= (-1)^{N+1}\mathbf{c}\varphi(a_0,\dots,a_{N+1}) = (-1)^N\mathbf{b}'\varphi(a_0,\dots,a_{N+1})\\
&= (-1)^N\sum_{i=0}^{N}(-1)^i\varphi(a_0,\dots,a_ia_{i+1},\dots,a_{N+1})\\
&= (-1)^N\left(\sum_{i=1}^{N}(-1)^i\int' a_0\,\mathrm{d}a_1\cdots\mathrm{d}(a_ia_{i+1})\cdots\mathrm{d}a_{N+1} + \int' a_0a_1\,\mathrm{d}a_2\cdots\mathrm{d}a_{N+1}\right)\\
&= (-1)^N\left(\int' a_0((-1)^N(\mathrm{d}a_1\cdots\mathrm{d}a_N)a_{N+1} - a_1\,\mathrm{d}a_2\cdots\mathrm{d}a_{N+1})\right.\\
&\qquad\qquad \left.+\int' a_0a_1\,\mathrm{d}a_2\cdots\mathrm{d}a_{N+1}\right)\\
&= \int' a_0(\mathrm{d}a_1\cdots\mathrm{d}a_N)a_{N+1}.
\end{aligned}$$

In the other case

$$\begin{aligned}
\int' \sigma(a_{N+1})\,\mathrm{d}a_1\cdots\mathrm{d}a_N &= \varphi(\sigma(a_{N+1}),a_1,\dots,a_N) = (-1)^N\varphi(a_1,\dots,a_N,a_{N+1})\\
&= (-1)^N\int' a_1\mathrm{d}a_2\cdots\mathrm{d}a_{N+1} = \int'(\mathrm{d}a_1\cdots\mathrm{d}a_N)a_{N+1},
\end{aligned}$$

where we used the closeness of $\int'$ and the aforementioned fact in the last equality.

It follows now that $\int'$ is a twisted graded trace. Now let (Ω, d) be the N-dimensional differential calculus obtained from $\bar{\Omega}$ by truncation, and let $\int$ be the restriction of $\int'$ to Ω. Clearly, $\int$ is again a closed twisted graded trace and φ is the twisted cyclic cocycle associated to (Ω, d) and $\int$. □

If σ is an automorphism of a unital algebra $\mathcal{A}$ and $\varphi\in\mathbf{C}^*_\lambda(A,\sigma)$, we say that φ is *left faithful* if, for each element a in $\mathcal{A}$, we have $a = 0$, if $\varphi(aa_0,a_1,\dots,a_N) = 0$, for all $a_0,\dots,a_N\in\mathcal{A}$. Since $\lambda\varphi=\varphi$, we have, for each index $i = 0,\dots,N$, $a = 0$, if $\varphi(a_0,a_1,\dots,aa_i,\dots,a_N) = 0$, for all $a_0,\dots,a_N\in\mathcal{A}$.

Theorem 3.3. *Let σ be an automorphism of a unital algebra $\mathcal{A}$ and let $\varphi\in\mathbf{Z}^N_\lambda(\mathcal{A},\sigma)$, for some integer $N\geq 0$. If φ is left faithful, then there exists an N-dimensional differential* *calculus* (Ω, d) *over* $\mathcal{A}$ *and a left faithful* N*-dimensional, closed twisted graded trace* $\int$ *on* Ω *such that* φ *is the twisted cyclic cocycle associated to* (Ω, d) *and* $\int$.

Proof. Define an N-dimensional linear functional $\int'$ on the universal differential calculus $\bar{\Omega}$ over $\mathcal{A}$ by setting $\int' a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_N = \varphi(a_0, \dots, a_N)$ and $\int' \mathrm{d}a_1 \cdots \mathrm{d}a_N = 0$, for all $a_0, \dots, a_N \in \mathcal{A}$. We saw in the proof of the preceding theorem that $\int'$ is a closed twisted graded trace. The faithfulness assumption on φ ensures that $\int'$ is weakly faithful. Now let (Ω, d) be the N-dimensional calculus associated to $\bar{\Omega}$ and $\int'$ and let $\int$ be the canonical N-dimensional, left faithful, closed twisted graded trace on Ω. Then φ is clearly the twisted cyclic cocycle associated to $\int$. □

To round off this circle of ideas, let us note that if $\int$ is any N-dimensional, weakly faithful, closed twisted graded trace on a differential calculus (Ω, d) over a unital algebra $\mathcal{A}$, the associated twisted cyclic cocycle φ is clearly left faithful.

We turn now to the case of $*$-differential calculi. If (Ω, d) is such a calculus over a unital $*$-algebra $\mathcal{A}$, then it is readily verified that, for all 1-forms $\omega_1, \dots, \omega_N$ of Ω, $(\omega_1 \cdots \omega_N)^* = s_N \omega_N^* \cdots \omega_1^*$, where (s_N) is the sequence of scalars defined inductively by $s_1 = 1$ and $s_{N+1} = (-1)^N s_N$. If φ is the N-cocycle associated to an N-dimensional weakly faithful, closed, self-adjoint, twisted graded trace $\int$ on Ω, then $\varphi^* = \varphi$, where $\varphi^*(a_0, \dots, a_N) = s_{N+1}\bar{\varphi}(a_N^*, \dots, a_0^*)$ (as usual, $\bar{\varphi}$ is the complex conjugate function corresponding to φ, so that $\bar{\varphi}(x) = \overline{\varphi(x)}$). To see that $\varphi^* = \varphi$, observe that, if σ is a twist automorphism associated to $\int$, then

$$\begin{aligned}
\varphi^*(a_0, \dots, a_N) &= s_{N+1}\bar{\varphi}(a_N^*, \dots, a_0^*) = s_{N+1}(-1)^N \bar{\varphi}(\sigma(a_0^*), a_N^*, \dots, a_1^*) \\
&= s_{N+1}(-1)^N \left(\int \sigma(a_0^*)(\mathrm{d}a_N^*) \cdots (\mathrm{d}a_1^*) \right)^{-} \\
&= (-1)^N s_N s_{N+1} \int (\mathrm{d}a_1) \cdots (\mathrm{d}a_n) \sigma(a_0^*)^* \\
&= s_{N+1}^2 \int (\mathrm{d}a_1) \cdots (\mathrm{d}a_N) \sigma^{-1}(a_0) \\
&= \int a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_N = \varphi(a_0, \dots, a_N).
\end{aligned}$$

Here, in the third last equation, we have used the easily verified fact that $\sigma^{-1}(a^*) = \sigma(a)^*$, for all $a \in \mathcal{A}$ (this uses weak faithfulness of $\int$).

These observations motivate the following definitions.

If the function, $\varphi : \mathcal{A}^{N+1} \to \mathbf{C}$, is multilinear, we define φ^* by setting $\varphi^*(a_0, \dots, a_N) = s_{N+1}\bar{\varphi}(a_N^*, \dots, a_0^*)$, for all $a_0, \dots, a_N \in \mathcal{A}$.

If σ is an automorphism of $\mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$, for all $a \in \mathcal{A}$, then we call σ *regular*. As we observed above, the restriction to $\mathcal{A}$ of a twist automorphism associated to a weakly faithful, self-adjoint twisted graded trace is regular. Another observation: if σ is any self-adjoint automorphism of $\mathcal{A}$ and $\sigma^2 = \mathrm{id}$, then σ is regular.

It is easy check that, if σ is any regular automorphism of $\mathcal{A}$, and $\varphi \in \mathbf{C}_\lambda^N(\mathcal{A}, \sigma)$, then $\varphi^* \in \mathbf{C}_\lambda^N(\mathcal{A}, \sigma)$. It is also the case that, if $\mathbf{b}\varphi = 0$, then $\mathbf{b}\varphi^* = 0$. However, this requires some proof, so we give the details. It clearly suffices to show that, if $a_0, \dots, a_{N+1} \in \mathcal{A}$, then

$$\sum_{i=0}^{N}(-1)^i \varphi(a_{N+1}^*, \dots, a_{i+1}^* a_i^*, \dots, a_0^*) + (-1)^{N+1}\varphi(a_N^*, \dots, a_1^*, a_0^*\sigma(a_{N+1})^*) = 0.$$

Set $b_i = a_{N+1-i}^*$, for $i = 0, \dots, N+1$. Multiplying the above equation by $(-1)^N$ and using the fact that $\sigma(a_{N+1})^* = \sigma^{-1}(a_{N+1}^*) = \sigma^{-1}(b_0)$, we see that we need only show that

$$\sum_{i=0}^{N}(-1)^{N-i}\varphi(b_0, \dots, b_{N-i}b_{N-i+1}, \dots, b_{N+1}) \\ +(-1)^{2N+1}\varphi(b_1, \dots, b_N, b_{N+1}\sigma^{-1}(b_0)) = 0.$$

Now we use the fact that $\lambda\varphi = \varphi$, which implies that $(-1)^N\varphi(b_1, \dots, b_N, b_{N+1}\sigma^{-1}(b_0)) = \varphi(\sigma(b_{N+1})b_0, b_1, \dots, b_N)$, to see that we have only to show that

$$\sum_{i=0}^{N}(-1)^{N-i}\varphi(b_0, \dots, b_{N-i}b_{N-i+1}, \dots, b_{N+1}) \\ +(-1)^{N+1}\varphi(\sigma(b_{N+1})b_0, b_1, \dots, b_N) = 0;$$

that is, it suffices to show that

$$\sum_{i=0}^{N}(-1)^i\varphi(b_0, \dots, b_i b_{i+1}, \dots, b_{N+1}) + (-1)^{N+1}\varphi(\sigma(b_{N+1})b_0, b_1, \dots, b_N) = 0.$$

However, this is true, since it is just the equation $(\mathbf{b}' + \mathbf{c})\varphi(b_0, \dots, b_{N+1}) = 0$, which holds because $\mathbf{b}\varphi = 0$, by assumption.

If we define φ to be *self-adjoint*, if $\varphi^* = \varphi$, then the preceding observations, together with the easily checked equation $(\varphi^*)^* = \varphi$, show that every element $\varphi \in \mathbf{Z}_\lambda^N(\mathcal{A}, \sigma)$ can be written in the form $\varphi = \varphi_1 + i\varphi_2$, for some self-adjoint elements φ_1 and φ_2 in $\mathbf{Z}_\lambda^N(\mathcal{A}, \sigma)$. (Of course, one sets $\varphi_1 = (\varphi + \varphi^*)/2$ and $\varphi_2 = (\varphi - \varphi^*)/2i$.)

Now suppose that $\int$ is an N-dimensional, closed, twisted graded trace on a $*$-differential calculus (Ω, d). If the twisted cyclic N-cocycle φ associated to $\int$ is self-adjoint, then $\int$ is self-adjoint. To see this we need only show that $(\int \omega)^- = \int \omega^*$, where $\omega = a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_N$ or $\omega = \mathrm{d}a_1 \cdots \mathrm{d}a_N$, for elements $a_0, \dots, a_N$ belonging to $\mathcal{A}$. However, we have

$$\begin{aligned}\left(\int \omega\right)^- &= \bar{\varphi}(a_0, \dots, a_N) = \bar{\varphi}^*(a_0, \dots, a_N) = s_{N+1}\varphi(a_N^*, \dots, a_0^*) \\ &= s_{N+1}\int a_N^*(\mathrm{d}a_1^*) \cdots (\mathrm{d}a_0^*) = (-1)^N \int ((\mathrm{d}a_0) \cdots (\mathrm{d}a_{N-1})a_N)^* \\ &= (-1)^N \int (\mathrm{d}(a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{N-1})a_N)^* = \int (a_0\, \mathrm{d}(a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_N))^* = \int \omega^*.\end{aligned}$$

In the second last equation we used the fact that $\int d = 0$ and that $d((a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{N-1})a_N) = d(a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{N-1})a_N + (-1)^{N-1}a_0 d(a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_N)$.

If $\omega = \mathrm{d}a_1 \cdots \mathrm{d}a_N$, it is clear that $\int \omega = 0 = \int \omega^*$ due to the closeness of $\int$.

We sum up our observations in the following theorem.

Theorem 3.4. *Let $\mathcal{A}$ be a unital $*$-algebra and let σ be a regular (algebra) automorphism of $\mathcal{A}$. Let $\int$ be an N-dimensional, closed, twisted graded trace on a $*$-differential calculus (Ω, d) over $\mathcal{A}$, and suppose that its twist automorphism extends σ. Let φ be the twisted cyclic N-cocycle associated to $\int$, so that $\varphi \in \mathbf{Z}_\lambda^N(\mathcal{A}, \sigma)$. Then φ is self-adjoint if, and only if, $\int$ is self-adjoint.*

4. Twisted cyclic cohomology

In this section, we briefly consider the twisted cyclic cohomology theory of a pair $(\mathcal{A}, \sigma)$, where $\mathcal{A}$ is a unital algebra and σ is an automorphism of $\mathcal{A}$. We shall be particularly interested in the construction of analogues of the important operators $\mathbf{S}$ and $\mathbf{B}$ occurring in the classical cyclic cohomology theory. These are used to relate twisted cyclic cohomology to twisted Hochschild cohomology. We begin by defining the latter. Note that if $\varphi \in \mathbf{C}^n(\mathcal{A})$, then $(\lambda^{n+1}\varphi)(a_0, \ldots, a_n) = \varphi(\sigma(a_0), \ldots, \sigma(a_n))$, for all $a_0, \ldots, a_n \in \mathcal{A}$. Let $\mathbf{C}^*(\mathcal{A}, \sigma) = \oplus_{n\in\mathbf{N}}\mathbf{C}^n(\mathcal{A}, \sigma)$, where $\mathbf{C}^n(\mathcal{A}, \sigma) = \{\varphi \in \mathbf{C}^n(\mathcal{A}) | \lambda^{n+1}\varphi = \varphi\}$. One can show that, for $\varphi \in \mathbf{C}^n(\mathcal{A})$, we have $\mathbf{b}\lambda^{n+1}\varphi = \lambda^{n+2}\mathbf{b}\varphi$ and $\mathbf{b}'\lambda^{n+1}\varphi = \lambda^{n+2}\mathbf{b}'\varphi$. It follows that $\mathbf{C}^*(\mathcal{A}, \sigma)$ is invariant for $\mathbf{b}$ and $\mathbf{b}'$ and therefore we get a cochain complex $(\mathbf{C}^*(\mathcal{A}, \sigma), \mathbf{b})$. We denote its cohomology by $\mathbf{HH}(\mathcal{A}, \sigma)$ and call it the *twisted Hochschild cohomology* of the pair $(\mathcal{A}, \sigma)$.

We shall now get the twisted cyclic cohomology as the cohomology of the total complex of a bicomplex. To define this bicomplex we introduce the operator $\mathbf{N}$ of degree zero on $\mathbf{C}^*(\mathcal{A}, \sigma)$, defined, for $\varphi \in \mathbf{C}^n(\mathcal{A}, \sigma)$, by setting $\mathbf{N}\varphi = \sum_{i=0}^n \lambda^i \varphi$. One can show that $\mathbf{bN} = \mathbf{Nb}'$ and $(1-\lambda)\mathbf{b} = \mathbf{b}'(1-\lambda)$ and $\mathbf{N}(1-\lambda) = 0$. Hence, for $\mathbf{C}^n = \mathbf{C}^n(\mathcal{A}, \sigma)$, the following diagram defines a bicomplex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \\
\mathbf{C}^2 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^2 & \stackrel{N}{\rightarrow} & \mathbf{C}^2 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^2 & \stackrel{\mathbf{N}}{\rightarrow} & \cdots \\
\mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \\
\mathbf{C}^1 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^1 & \stackrel{\mathbf{N}}{\rightarrow} & \mathbf{C}^1 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^1 & \stackrel{\mathbf{N}}{\rightarrow} & \cdots \\
\mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \mathbf{b}\uparrow & & -\mathbf{b}'\uparrow & & \\
\mathbf{C}^0 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^0 & \stackrel{\mathbf{N}}{\rightarrow} & \mathbf{C}^0 & \stackrel{1-\lambda}{\rightarrow} & \mathbf{C}^0 & \stackrel{\mathbf{N}}{\rightarrow} & \cdots
\end{array}$$

We denote this bicomplex by $\mathbf{C}^{**}(\mathcal{A}, \sigma)$ and its total complex by $\mathbf{T}^*(\mathcal{A}, \sigma)$. The entry in the bicomplex at the position (m, n) is $\mathbf{C}^{m,n}(\mathcal{A}, \sigma) = \mathbf{C}^n(\mathcal{A}, \sigma)$. We denote the cohomology of $\mathbf{T}^*(\mathcal{A}, \sigma)$ by $\mathbf{HC}^*(\mathcal{A}, \sigma)$. We shall see that this is isomorphic to $\mathbf{H}_\lambda(\mathcal{A}, \sigma)$. The advantage of this alternative description is that it enables us to define the operators $\mathbf{S}$ and $\mathbf{B}$ in a natural way.

We define a cochain map π from the complex $\mathbf{C}_\lambda^*(\mathcal{A}, \sigma)$ to the complex $\mathbf{T}^*(\mathcal{A}, \sigma)$ by mapping x in $\mathbf{C}_\lambda^n(\mathcal{A}, \sigma)$ onto $(x, 0, \ldots, 0)$ in $\mathbf{T}^n(\mathcal{A}, \sigma) = \oplus_{i=0}^n \mathbf{C}^{i,n-i}(\mathcal{A}, \sigma)$. Then one can show that the induced linear map, $\pi_* : \mathbf{H}_\lambda^*(\mathcal{A}, \sigma) \to \mathbf{HC}^*(\mathcal{A}, \sigma)$, is an isomorphism.

We now define $\mathbf{C}^{**}_{[2]}$ to be the cochain bicomplex obtained from $\mathbf{C}^{**}(\mathcal{A},\sigma)$ by restricting to the first two columns and setting all other columns equal to zero. Let $\mathbf{T}^{*}_{[2]}(\mathcal{A},\sigma)$ be the total complex of $\mathbf{C}^{**}_{[2]}$. We define a cochain map θ from $\mathbf{T}^{*}_{[2]}(\mathcal{A},\sigma)$ to $\mathbf{C}^{*}(\mathcal{A},\sigma)$ by setting $\theta(x)=x$, for x in $\mathbf{T}^{0}_{[2]}(\mathcal{A},\sigma)=\mathbf{C}^{0}(\mathcal{A},\sigma)$ and setting $\theta(x_0,x_1)=x_0$, for (x_0,x_1) in $\mathbf{T}^{n}_{[2]}(\mathcal{A},\sigma)=\mathbf{C}^{n}(\mathcal{A},\sigma)\oplus\mathbf{C}^{n-1}(\mathcal{A},\sigma)$, where $n>0$. The induced map θ_* mapping $\mathbf{H}^*(\mathbf{T}^{*}_{[2]}(\mathcal{A},\sigma))$ to $\mathbf{HH}^*(\mathcal{A},\sigma)$, is an isomorphism.

Now we define a cochain map of degree 2 on $\mathbf{T}^*(\mathcal{A},\sigma)$ by shifting its chain bicomplex two columns to the right; more precisely, if $x=(x_0,\dots,x_n)\in\mathbf{T}^n(\mathcal{A},\sigma)$, set $\mathbf{R}(x)=(0,0,x_0,\dots,x_n)$. Let $\mathbf{P}$ be the degree zero cochain map from $\mathbf{T}^*(\mathcal{A},\sigma)$ to $\mathbf{T}^{*}_{[2]}(\mathcal{A},\sigma)$ obtained by projecting; more precisely, $\mathbf{P}(x)=x$ for $x\in\mathbf{T}^0(\mathcal{A},\sigma)$ and $\mathbf{P}(x)=(x_0,x_1)$, for $x=(x_0,\dots,x_n)\in\mathbf{T}^n(\mathcal{A},\sigma)$, where $n>0$. This gives a short exact sequence of cochain maps

$$0\to\mathbf{T}^*(\mathcal{A},\sigma)\overset{\mathbf{R}}{\to}\mathbf{T}^*(\mathcal{A},\sigma)\overset{\mathbf{P}}{\to}\mathbf{T}^{*}_{[2]}(\mathcal{A},\sigma)\to 0.$$

On the cohomological level we therefore get an exact triangle

$$\begin{array}{ccc}\mathrm{H}^*(\mathrm{T}^*_{[2]}(\mathcal{A},\sigma)) & \overset{\partial}{\longrightarrow} & \mathrm{H}^*(\mathrm{T}^*(\mathcal{A},\sigma))\\ \mathrm{P}_*\nwarrow & & \swarrow\mathrm{R}_*\\ & \mathrm{H}^*(\mathrm{T}^*(\mathcal{A},\sigma)) & \end{array}$$

Finally, we define the linear maps $\mathbf{I}:\mathbf{H}^*_\lambda(\mathcal{A},\sigma)\to\mathbf{HH}^*(\mathcal{A},\sigma)$, $\mathbf{S}:\mathbf{H}^*_\lambda(\mathcal{A},\sigma)\to\mathbf{H}^*_\lambda(\mathcal{A},\sigma)$ and $\mathbf{B}:\mathbf{HH}^*(\mathcal{A},\sigma)\to\mathbf{H}^*_\lambda(\mathcal{A},\sigma)$ of degrees 0, 2 and -1 respectively by setting $\mathbf{I}=\theta_*\mathbf{P}_*\pi_*$, $\mathbf{S}=\pi_*^{-1}\mathbf{R}_*\pi_*$ and $\mathbf{B}=\pi_*^{-1}\partial\theta_*^{-1}$. This gives us an exact triangle

$$\begin{array}{ccc}\mathrm{HH}^*(\mathcal{A},\sigma) & \overset{B}{\longrightarrow} & \mathrm{H}^*_\lambda(\mathcal{A},\sigma)\\ \mathrm{I}\nwarrow & & \swarrow\mathrm{S}\\ & \mathrm{H}^*_\lambda(\mathcal{A},\sigma) & \end{array}$$

By expansion of this we get a long exact sequence

$$\cdots\to\mathbf{H}^{n-2}_\lambda(\mathcal{A},\sigma)\overset{\mathbf{S}}{\to}\mathbf{H}^{n}_\lambda(\mathcal{A},\sigma)\overset{\mathbf{I}}{\to}\mathbf{HH}^{n}(\mathcal{A},\sigma)\overset{\mathbf{B}}{\to}\mathbf{H}^{n-1}_\lambda(\mathcal{A},\sigma)\overset{\mathbf{S}}{\to}\mathbf{H}^{n+1}_\lambda(\mathcal{A},\sigma)\to\cdots$$

Thus, we have indicated how the principal results of the elementary theory of cyclic cohomology extends to the twisted case. Since the proofs in this more general setting are essentially the same as in the non-twisted case, we have omitted the details.

5. Left-covariant differential calculi

Differential calculi that are left-covariant are of prime importance for the theory. We shall introduce this concept now. For this we need to suppose that $\mathcal{A}$ is endowed with a co-multiplication Δ making the pair $(\mathcal{A},\Delta)$ a Hopf algebra (such an algebra is unital by assumption). In the sequel we shall use a number of elementary results about Hopf algebras without explicit reference. A good general source for this material is [1].

Recall that a left-covariant bi-module over $\mathcal{A}$ d is a pair (Γ, Δ_Γ), where Γ is a bi-module over $\mathcal{A}$, and Δ_Γ is a linear map from Γ to $\mathcal{A} \otimes \Gamma$ such that the following conditions hold:

(1) $(\Delta \otimes \mathrm{id}_\Gamma)\Delta_\Gamma = (\mathrm{id}_\mathcal{A} \otimes \Delta_\Gamma)\Delta_\Gamma$ and $(\mathrm{e} \otimes \mathrm{id}_\Gamma)\Delta_\Gamma = \mathrm{id}_\Gamma$, where e is the co-unit of $(\mathcal{A}, \Delta)$, (that is, Δ_Γ is a left coaction);
(2) $\Delta_\Gamma(a\gamma b) = \Delta(a)\Delta_\Gamma(\gamma)\Delta(b)$, for all $\gamma \in \Gamma$ and $a, b \in \mathcal{A}$.

Note that (Γ, Δ_Γ) is a left $\mathcal{A}$-comodule (see e.g. [4, 1.3.2 Definition 7]). Later on, we will use the Sweedler notation for such left comodules as explained in [4, 1.3.2 Eq. (60)].

An element $\gamma \in \Gamma$ is said to be *left-invariant* if $\Delta_\Gamma(\gamma) = 1 \otimes \gamma$. We denote by Γ^{inv} the linear space of left-invariant elements of Γ.

If $a \in \mathcal{A}$ and f is a linear functional on $\mathcal{A}$, we set $f * a = (\mathrm{id}_\mathcal{A} \otimes f)\Delta(a)$. We shall make use of the following result from the theory of left-covariant bi-modules.

Theorem 5.1 (S.L. Woronowicz [8,10]). *Let (Γ, Δ_Γ) be a left-covariant bi-module over a Hopf algebra $(\mathcal{A}, \Delta)$.*

(1) *There is a unique isomorphism of left $\mathcal{A}$-modules from $\mathcal{A} \otimes \Gamma^{\mathrm{inv}}$ onto Γ that maps $a \otimes \gamma$ onto $a\gamma$, for all $a \in \mathcal{A}$ and $\gamma \in \Gamma^{\mathrm{inv}}$.*
(2) *Suppose that the family of elements $(\gamma_i)_{i\in I}$ is a linear basis for Γ^{inv}. Then it is a free left $\mathcal{A}$-module basis for Γ and also a free right $\mathcal{A}$-module basis of Γ. Moreover, there exist linear functionals f_{jk} on $\mathcal{A}$, for all $j, k \in I$, such that $f_{jk}(ab) = \sum_{i\in I} f_{ji}(a) f_{ik}(b)$ and $f_{jk}(1) = \delta_{jk}$ and for which we have the equations $\gamma_j a = \sum_{i\in I}(f_{ji} * a)\gamma_i$ and $a\gamma_j = \sum_{i\in I}\gamma_i((f_{ji}\kappa^{-1}) * a)$, where κ is the co-inverse for $(\mathcal{A}, \Delta)$.*

When we consider a sum $\sum_{i\in I} x_i$ of a family $(x_i)_{i\in I}$ of elements in a vector space X with no topological structure, it is understood that $x_i = 0$ for all but a finite number of indices $i \in I$.

Let (Ω, d) be a unital differential calculus over $\mathcal{A}$ such that $d1 = 0$. This is a bi-module over $\mathcal{A}$ in a natural way. If the map, $\Delta_\Omega : \Omega \to \mathcal{A} \otimes \Omega$, makes Ω into a left-covariant bi-module and $(\mathrm{id}_\mathcal{A} \otimes d)\Delta_\Omega = \Delta_\Omega d$, and $\Delta_\Omega(a) = \Delta(a)$, for all $a \in \mathcal{A}$, we call the triple $(\Omega, d, \Delta_\Omega)$ a *left-covariant differential calculus* over $(\mathcal{A}, \Delta)$. A moment's reflection, using the fact that Ω is generated as an algebra by the elements a and da, where $a \in \mathcal{A}$, shows that only one such left action Δ_Ω can exist making $(\Omega, d, \Delta_\Omega)$ a left-covariant calculus. For this reason, we often speak of the left-covariant differential calculus (Ω, d), omitting explicit reference to Δ_Ω. Henceforth, we shall also often speak of the Hopf algebra $\mathcal{A}$, omitting explicit reference of the co-multiplication Δ.

The map Δ_Ω is automatically of degree zero, where we regard $\mathcal{A} \otimes \Omega$ as graded algebra in the obvious way (its space of k-forms is the tensor product $\mathcal{A} \otimes \Omega_k$).

The linear span of the set $\Delta(\mathcal{A})(\mathcal{A}\otimes 1) = \{\Delta(a)(b\otimes 1)|a, b \in \mathcal{A}\}$ is equal to $\mathcal{A}\otimes\mathcal{A}$ (this is true for any Hopf algebra). It follows from this that the linear span of $\Delta_\Omega(\Omega)(\mathcal{A} \otimes 1)$ is equal to $\mathcal{A} \otimes \Omega$.

We shall denote the linear space of left-invariant k-forms of Ω by Ω_k^{inv}.

Let $\mathcal{A}$ be any unital algebra (not necessarily the underlying algebra of a Hopf algebra). In Section 2 we introduced the universal differential algebra $(\bar{\Omega}, d)$ over $\mathcal{A}$ (which is not unital). But there also exists a universal unital differential algebra over $\mathcal{A}$ and this is the one we will be working with in the rest of this paper. There exists a unital graded differential algebra $(\tilde{\Omega}, d)$, for which $\tilde{\Omega}_0 = \mathcal{A}$, that has the following universal property: If σ is a unital algebra homomorphism from $\mathcal{A}$ into the algebra Ω_0 of 0-forms of a unital graded differential algebra (Ω, d), then there exists a unique unital algebra homomorphism $\tilde{\sigma}$ from $\tilde{\Omega}$ to Ω extending σ such that $\tilde{\sigma} d = d\tilde{\sigma}$. This property uniquely determines $(\tilde{\Omega}, d)$ (up to isomorphism). Note that $d1 = 0$.

We shall use the following useful property of $(\tilde{\Omega}, d)$:

Let n be a non-negative integer and T a multilinear map from $\mathcal{A}^{n+1}$ to a linear space Y such that $T(a_0, \dots, a_n) = 0$, if any of the elements $a_1, \dots, a_n$ is a scalar. Then there is a unique linear map $\hat{T}$ from $\tilde{\Omega}_n$ to Y for which $\hat{T}(a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_n) = T(a_0, a_1, \dots, a_n)$, for all $a_0, a_1, \dots, a_n \in \mathcal{A}$.

Theorem 2.1 remains valid for $(\tilde{\Omega}, d)$ in place of $(\bar{\Omega}, d)$, provided σ_0 is assumed to be unital.

If $(\mathcal{A}, \Delta)$ is a Hopf algebra, then the universal unital calculus $(\tilde{\Omega}, d)$ over $\mathcal{A}$ is a left-covariant calculus over $(\mathcal{A}, \Delta)$. To see this, first observe that $\mathcal{A} \otimes \tilde{\Omega}$ can be made into a differential calculus, where $\mathrm{id}_{\mathcal{A}} \otimes d$ is its differential. The map Δ, regarded as an algebra homomorphism from $\mathcal{A}$ to the 0-forms of $\mathcal{A} \otimes \tilde{\Omega}$, extends to an algebra homomorphism Δ' from $\tilde{\Omega}$ to $\mathcal{A} \otimes \tilde{\Omega}$ such that $\Delta' d = (\mathrm{id}_{\mathcal{A}} \otimes d)\Delta'$. It now follows from the next lemma that $(\tilde{\Delta}, d, \Delta')$ is a left-covariant differential calculus over $(\mathcal{A}, \Delta)$.

Lemma 5.2. *Let (Ω, d) be a unital differential calculus over a Hopf algebra $(\mathcal{A}, \Delta)$ such that $d1 = 0$ and suppose that $\Delta_\Omega : \Omega \to \mathcal{A} \otimes \Omega$ is an algebra homomorphism extending $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $(\mathrm{id}_{\mathcal{A}} \otimes d)\Delta_\Omega = \Delta_\Omega d$. Then $(\Omega, d, \Delta_\Omega)$ is a left-covariant differential calculus.*

Proof. We have to prove that $(\Delta \otimes \mathrm{id}_\Omega)\Delta_\Omega = (\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega)\Delta_\Omega$ and $(\mathrm{e} \otimes \mathrm{id}_\Omega)\Delta_\Omega = \mathrm{id}_\Omega$, where e is the co-unit of $(\mathcal{A}, \Delta)$. We shall prove only the first of these equations; the proof of the second is straightforward. Since $(\Delta \otimes \mathrm{id}_\Omega)\Delta_\Omega$ and $(\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega)\Delta_\Omega$ are homomorphisms and Ω is generated as an algebra by the forms a and $\mathrm{d}a$, where $a \in \mathcal{A}$, we need only see that these homomorphisms are equal at such forms. This is obvious in the case of the elements a, since $\Delta_\Omega(a) = \Delta(a)$. For $\mathrm{d}a$ we have

$$
\begin{aligned}
(\Delta \otimes \mathrm{id}_\Omega)\Delta_\Omega\, \mathrm{d}(a) &= (\Delta \otimes \mathrm{id}_\Omega)(\mathrm{id}_{\mathcal{A}} \otimes \mathrm{d})\Delta(a) = (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes d)(\Delta \otimes \mathrm{id}_{\mathcal{A}})\Delta(a) \\
&= (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes d)(\mathrm{id}_{\mathcal{A}} \otimes \Delta)\Delta(a) = (\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega d)\Delta(a) \\
&= (\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega)(\mathrm{id}_{\mathcal{A}} \otimes d)\Delta(a) = (\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega)\Delta_\Omega d(a).
\end{aligned}
$$

This proves the lemma. □

Recall that a linear functional h on a Hopf algebra $\mathcal{A}$ is said to be *left-invariant* if $(\mathrm{id} \otimes h)\Delta(a) = h(a)1$, for all $a \in \mathcal{A}$, where 1 is the unit of $\mathcal{A}$. Similarly, a linear functional h' on $\mathcal{A}$ is *right-invariant* if $(h' \otimes \mathrm{id})\Delta(a) = h'(a)1$, for all $a \in \mathcal{A}$. Such functionals do not necessarily exist. It is easily seen that there is at most one unital linear functional h on $\mathcal{A}$ that is both left and right-invariant. We call such a functional a *Haar integral* of $\mathcal{A}$. In the sequel, we shall be principally interested in working with Hopf algebras that admit Haar integrals. If $\mathcal{A}$ is the Hopf algebra associated to a compact quantum group in the sense of Woronowicz, then it admits a Haar integral. From the point of view of relevance of the theory we are developing here, the Hopf algebras associated to quantum groups are those of prime interest.

We say that a linear functional $\int$ on a left-covariant differential calculus (Ω, d) over a Hopf algebra $\mathcal{A}$ is *left-invariant* if $(\mathrm{id}_{\mathcal{A}} \otimes \int)\Delta_\Omega(\omega) = (\int \omega)1$, for all $\omega \in \Omega$, where 1 is the unit of $\mathcal{A}$.

Clearly, the restriction of $\int$ to $\mathcal{A}$ is a left-invariant linear functional on $\mathcal{A}$; however, it may be equal to zero on $\mathcal{A}$ (this is frequently the case).

Theorem 5.3. *Let $\int$ be a linear functional on a left-covariant differential calculus (Ω, d) over a Hopf algebra $\mathcal{A}$. Suppose also that $\mathcal{A}$ admits a Haar integral h. Then the following are equivalent conditions:*

(1) $\int a\omega = h(a) \int \omega$, *for all $a \in \mathcal{A}$ and for all $\omega \in \Omega^{\mathrm{inv}}$;*
(2) $\int$ *is left-invariant.*

Proof. Assume first that $\int$ is left-invariant and suppose that $a \in \mathcal{A}$ and $\omega \in \Omega^{\mathrm{inv}}$. Since $h(1) = 1$, we have $\int a\omega = h((\int a\omega)1) = h((\mathrm{id}_{\mathcal{A}} \otimes \int)\Delta_\Omega(a\omega)) = (h \otimes \int)(\Delta(a)(1 \otimes \omega)) = \int((h \otimes \mathrm{id}_{\mathcal{A}})\Delta(a))\omega = \int h(a)\omega = h(a) \int \omega$. Hence, Condition (2) implies Condition (1).

Now suppose that Condition (1) holds, and let a and ω be as before. We may write $\Delta(a) = \sum_{i=1}^{M} b_i \otimes c_i$, for some elements b_i and c_i in $\mathcal{A}$. Then $(\mathrm{id}_{\mathcal{A}} \otimes \int)(\Delta_\Omega(a\omega)) = (\mathrm{id}_{\mathcal{A}} \otimes \int)(\Delta(a)(1 \otimes \omega)) = (\mathrm{id}_{\mathcal{A}} \otimes \int)(\sum_{i=1}^{M} b_i \otimes c_i\omega) = \sum_{i=1}^{M}(\int c_i\omega)b_i = \sum_{i=1}^{M} h(c_i)(\int \omega)b_i = (\mathrm{id}_{\mathcal{A}} \otimes h)(\Delta(a)) \int \omega = h(a)(\int \omega)1 = (\int a\omega)1$. Since Ω is the linear span of the elements $a\omega$, it follows that $\int$ is left-invariant. Hence, Condition (1) implies Condition (2). □

It is a well-known and useful result that if h is a left-invariant linear functional on a Hopf algebra $\mathcal{A}$ and κ is the co-inverse on $\mathcal{A}$, then

$$\kappa((\mathrm{id}_{\mathcal{A}} \otimes h)(\Delta(a)(1 \otimes b))) = (\mathrm{id}_{\mathcal{A}} \otimes h)((1 \otimes a)\Delta(b)),$$

for all elements $a, b \in \mathcal{A}$. We show now that a corresponding such result holds for left-invariant linear functionals on a differential calculus.

Theorem 5.4. *Let (Ω, d) be a left-covariant differential calculus over a Hopf algebra $\mathcal{A}$ and let $\int$ be a left-invariant linear functional on Ω. Then,*

$$\kappa\left(\left(\mathrm{id}_{\mathcal{A}} \otimes \int\right)(\Delta_\Omega(\omega)(1 \otimes \omega'))\right) = \left(\mathrm{id}_{\mathcal{A}} \otimes \int\right)((1 \otimes \omega)\Delta_\Omega(\omega')),$$

for all $\omega, \omega' \in \Omega$, where κ is the co-inverse of $\mathcal{A}$.

Proof. Using the Sweedler notation for left $\mathcal{A}$-comodules (see [4, 1.3.2 Eq. (60)]), we get that $\Delta_\Omega(\omega)(1 \otimes \omega') = \sum \omega_{(-1)} \otimes \omega_{(0)}\omega'$. Applying $\mathrm{id}_{\mathcal{A}} \otimes \Delta_\Omega$ to the right hand side of this equation, the left $\mathcal{A}$-comodule property of Ω guarantees that $\sum \omega_{(-1)} \otimes \Delta_\Omega(\omega_{(0)}\omega') = \sum \omega_{(-2)} \otimes \omega_{(-1)}\omega'_{(-1)} \otimes \omega_{(0)}\omega'_{(0)}$. If we apply $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes \int$ to this equation and use the left invariance of $\int$, we see that

$$\left(\mathrm{id}_{\mathcal{A}} \otimes \int\right) (\Delta_\Omega(\omega)(1 \otimes \omega')) \otimes 1$$
$$= \left(\sum \omega_{(-1)} \int \omega_{(0)}\omega'\right) \otimes 1 = \sum \omega_{(-2)} \otimes \omega_{(-1)}\omega'_{(-1)} \int \omega_{(0)}\omega'_{(0)}.$$

By applying $m(\kappa \otimes \mathrm{id}_{\mathcal{A}})$ to this equation and using the equalities $m(\kappa \otimes \mathrm{id}_{\mathcal{A}})\Delta = \mathrm{e}(\cdot)\,1$ and $(\mathrm{e} \otimes \mathrm{id}_\Omega)\Delta_\Omega = \mathrm{id}_\Omega$, this implies

$$\kappa\left(\left(\mathrm{id}_{\mathcal{A}} \otimes \int\right) (\Delta_\Omega(\omega)(1 \otimes \omega'))\right)$$
$$= \sum \kappa(\omega_{(-2)})\omega_{(-1)}\omega'_{(-1)} \int \omega_{(0)}\omega'_{(0)} = \sum \mathrm{e}(\omega_{(-1)})\omega'_{(-1)} \int \omega_{(0)}\omega'_{(0)}$$
$$= \sum \omega'_{(-1)} \int \mathrm{e}(\omega_{(-1)})\omega_{(0)}\omega'_{(0)} = \sum \omega'_{(-1)} \int \omega\omega'_{(0)}$$
$$= \left(\mathrm{id}_{\mathcal{A}} \otimes \int\right) ((1 \otimes \omega)\Delta_\Omega(\omega')). \qquad \square$$

Theorem 5.5. *Let (Ω, d) be a left-covariant differential calculus over a Hopf algebra $\mathcal{A}$ admitting a Haar integral h. Then the linear map, $P : \Omega \to \Omega$, defined by setting $P = (h \otimes \mathrm{id}_\Omega)\Delta_\Omega$, is idempotent with image equal to Ω^{inv}; also, $P(\omega_1\omega\omega_2) = \omega_1 P(\omega)\omega_2$, for all $\omega \in \Omega$ and $\omega_1, \omega_2 \in \Omega^{\mathrm{inv}}$. Moreover, $Pd = dP$. If $\int$ is a left-invariant linear functional on Ω, then $\int P(\omega) = \int \omega$, for all $\omega \in \Omega$.*

Proof. It is clear that $P(\omega) = \omega$ for all $\omega \in \Omega^{\mathrm{inv}}$. If ω in Ω, then $P(\omega) = \sum h(\omega_{(-1)})\omega_{(0)}$. Hence, using the left invariance of h in the second equality, we see that

$$\Delta_\Omega(P(\omega)) = \sum h(\omega_{(-2)})\omega_{(-1)} \otimes \omega_{(0)} = \sum 1 \otimes h(\omega_{(-1)})\omega_{(0)} = 1 \otimes P(\omega),$$

hence $P(\omega) \in \Omega^{\mathrm{inv}}$. It follows that $P^2 = P$ and $P(\Omega) = \Omega^{\mathrm{inv}}$.

Now suppose that ω is an arbitrary form of Ω and that $\omega_1, \omega_2 \in \Omega^{\mathrm{inv}}$. Then $P(\omega_1\omega\omega_2) = (h \otimes \mathrm{id}_\Omega)((1 \otimes \omega_1)\Delta_\Omega(\omega)(1 \otimes \omega_2)) = \omega_1(h \otimes \mathrm{id}_\Omega)(\Delta_\Omega(\omega))\omega_2 = \omega_1 P(\omega)\omega_2$.

We also have $Pd(\omega) = (h \otimes \mathrm{id}_\Omega)\Delta_\Omega d(\omega) = (h \otimes \mathrm{id}_\Omega)(\mathrm{id}_{\mathcal{A}} \otimes d)\Delta_\Omega(\omega) = d(h \otimes \mathrm{id}_\Omega)\Delta_\Omega(\omega) = dP(\omega)$. Hence, $Pd = dP$.

Suppose now $\int$ is a left-invariant linear functional on Ω. Then $\int P(\omega) = \int(h \otimes \mathrm{id}_{\mathcal{A}})\Delta_\Omega(\omega) = (h \otimes \int)\Delta_\Omega(\omega) = h((\mathrm{id}_{\mathcal{A}} \otimes \int)\Delta_\Omega(\omega)) = h((\int \omega)1) = \int \omega$. $\square$

If ω' and ω are invariant elements of Ω, then $\int \omega' a\omega = h(a) \int \omega'\omega$, since $\int \omega' a\omega = \int P(\omega' a\omega) = \int \omega' P(a)\omega = h(a) \int \omega'\omega$.

Corollary 5.6. *The linear space of N-dimensional, left-invariant linear functionals on Ω is linearly isomorphic to the linear dual of Ω_N^{inv}. Hence, Ω admits a unique non-zero,*

N-dimensional, left-invariant linear functional, up to a non-zero scalar factor, if, and only if, $\dim(\Omega_N^{\mathrm{inv}}) = 1$.

Proof. It follows directly from the theorem that he restriction map, $\int \mapsto \int_{\Omega_N^{\mathrm{inv}}}$, is the linear isomorphism of the preceding statement. Surjectivity of this map is the only non-obvious point. This is seen by observing that if τ is a linear functional on Ω_N^{inv}, then we can define the corresponding linear functional on Ω by setting $\int \omega = 0$, if ω is a k-form for which $k < N$, and by setting $\int \omega = \tau P(\omega)$, if $\omega \in \Omega_N$. Then if $a \in \mathcal{A}$ and $\omega \in \Omega_N^{\mathrm{inv}}$, and if $\Delta(a) = \sum_{i=1}^M b_i \otimes c_i$, for some elements b_i and c_i belonging to $\mathcal{A}$, we have $(\mathrm{id}_{\mathcal{A}} \otimes \int)(\Delta_\Omega(a\omega)) = (\mathrm{id}_{\mathcal{A}} \otimes \int)(\Delta(a)(1 \otimes \omega)) = \sum_{i=1}^M \tau(P(c_i)\omega) b_i = \sum_{i=1}^M h(c_i)\tau(\omega) b_i = (\mathrm{id}_{\mathcal{A}} \otimes h)(\Delta(a))\tau(\omega) = h(a)\tau(\omega)1 = \tau P(a\omega) = (\int a\omega)1$. Hence, by Theorem 5.1, $\int$ is left-invariant. □

A Haar integral h on a Hopf algebra $\mathcal{A}$ is necessarily *left faithful* in the sense that, whenever a is an element of $\mathcal{A}$ for which $h(ba) = 0$, for all $b \in \mathcal{A}$, we must have $a = 0$.

Theorem 5.7. *Let $\int$ be a non-zero, left-invariant linear functional on a left-covariant differential calculus (Ω, d) over a Hopf algebra $\mathcal{A}$ admitting a Haar integral h. Then $\int$ is weakly faithful.*

Proof. Suppose that $a \in \mathcal{A}$ and that $\int \omega a = 0$, for all $\omega \in \Omega$. Since $\int \neq 0$, we may choose ω such that $\int \omega \neq 0$. Then, for all $b \in \mathcal{A}$, we have $0 = \int \omega ba = \int P(\omega ba) = \int P(\omega) h(ba) = (\int \omega) h(ba)$. It follows, from faithfulness of h, that $a = 0$. Hence, $\int$ is weakly faithful. □

Theorem 5.8. *Let (Ω, d) be an N-dimensional left-covariant differential calculus over the Hopf algebra $\mathcal{A}$ admitting a Haar integral h. If (Ω, d) admits a left faithful, left-invariant, N-dimensional linear functional $\int$, then* $\dim(\Omega_N^{\mathrm{inv}}) = 1$.

Proof. Let ω be an invariant N-form of Ω for which $\int \omega = 0$. If $a \in \mathcal{A}$, then $\int a\omega = h(a) \int \omega = 0$. It follows, by faithfulness of $\int$, that $\omega = 0$. Therefore, the linear map, $\int : \Omega_N^{\mathrm{inv}} \to \mathbf{C}$, is injective. Since $\int$ is non-zero and left-invariant, this restriction map cannot be the zero map. Hence, it is a linear isomorphism of Ω_N^{inv} onto $\mathbf{C}$. Therefore, $\dim(\Omega_N^{\mathrm{inv}}) = 1$, as required. □

Corollary 5.9. *The functional $\int$ is closed if, and only if, $d(\Omega_{N-1}^{\mathrm{inv}}) = 0$. If $\int$ is closed, it is necessarily a twisted graded trace.*

Proof. First observe that if $P = (h \otimes \mathrm{id}_\Omega)\Delta_\Omega$, and $a \in \mathcal{A}$ and $\omega \in \Omega^{\mathrm{inv}}$, then $\int (\mathrm{d}a)\omega = \int P((\mathrm{d}a)\omega) = \int P(\mathrm{d}a)\omega = \int (dP(a))\omega = 0$, since $P(a) \in \mathbf{C}1$ and $d1 = 0$. Hence, $\int d(a\omega) = \int a\, \mathrm{d}\omega + \int (\mathrm{d}a)\omega = \int a\, \mathrm{d}\omega$. Using the identification $\Omega_{N-1} = \mathcal{A}\Omega_{N-1}^{\mathrm{inv}}$, it follows from this observation that if $d(\Omega_{N-1}^{\mathrm{inv}}) = 0$, then $\int d = 0$; that is, $\int$ is closed. Suppose now conversely that $\int$ is closed and let $\omega \in \Omega_{N-1}^{\mathrm{inv}}$. Then $0 = \int d(a\omega) = \int a\, \mathrm{d}\omega$, for all $a \in \mathcal{A}$. By faithfulness of $\int$, $d(\omega) = 0$. Hence, $d(\Omega_{N-1}^{\mathrm{inv}}) = 0$, as required.

Now suppose that $\int$ is closed and we shall show it is a twisted graded trace. Choose any non-zero element θ in Ω_N^{inv} for which $\int \theta = 1$; then $\Omega_N^{\text{inv}} = \mathbf{C}\theta$. Since $\mathcal{A}\theta = \theta\mathcal{A}$, by Theorem 5.1, there is a unique automorphism ρ_1 of $\mathcal{A}$ such that $\theta a = \rho_1(a)\theta$, for all $a \in \mathcal{A}$. Also, the Haar integral h admits another automorphism ρ_2 of $\mathcal{A}$ such that $h(ba) = h(\rho_2(a)b)$, for all $a, b \in \mathcal{A}$. Set $\sigma_0 = \rho_2\rho_1$. Then $\int b\theta a = \int b\rho_1(a)\theta = h(b\rho_1(a)) = h(\rho_2\rho_1(a)b) = \int \sigma_0(a)b\theta$. It follows from Theorem 2.2 that $\int$ is a twisted graded trace. □

We say that an N-dimensional differential calculus (Ω, d) over a unital algebra $\mathcal{A}$ is *non-degenerate* if, whenever ω is a k-form in Ω for which $\omega'\omega = 0$, for all $\omega' \in \Omega_{N-k}$, we necessarily have $\omega = 0$. It is clear that if Ω admits a left faithful, N-dimensional linear functional, then Ω is non-degenerate.

Theorem 5.10. *Let (Ω, d) be a non-degenerate, N-dimensional, left-covariant differential calculus over a Hopf algebra $\mathcal{A}$ admitting a Haar integral h. Then Ω admits a left faithful, left-invariant, N-dimensional linear functional if, and only if,* $\dim(\Omega_N^{\text{inv}}) = 1$.

Proof. The forward implication follows from Corollary 5.6. Suppose conversely $\dim(\Omega_N^{\text{inv}}) = 1$. Then, by Corollary 5.6, Ω admits a non-zero, N-dimensional, left-invariant linear functional $\int$ (unique up to multiplication by a scalar factor). To prove the theorem, we have only to show now that $\int$ is left faithful. Thus, we must show that if $\omega \in \Omega$ and $\int \omega'\omega = 0$, for all $\omega' \in \Omega$, then $\omega = 0$. We may clearly suppose, without loss of generality, that $\omega \in \Omega_k$, for some index $k \leq N$. Then if $\omega' \in \Omega_{N-k}$, we have $\omega'\omega = a\theta$, for some element $a \in \mathcal{A}$. Hence, if $b \in \mathcal{A}$, $\int b\omega'\omega = 0$, by assumption. Hence, $h(ba) = 0$, for all $b \in \mathcal{A}$. By faithfulness of h, $a = 0$. Therefore, $\omega'\omega = 0$. We now use non-degeneracy of Ω to deduce that $\omega = 0$, as required. □

Woronowicz has constructed a certain non-degenerate, left-covariant, three-dimensional calculus (Ω, d) over the Hopf algebra $\mathcal{A}$ underlying the compact quantum group $\mathrm{SU}_q(2)$, where q is a real parameter for which $0 < |q| \leq 1$. For this calculus, Ω_1^{inv} has a linear basis $\omega_0, \omega_1, \omega_2$ for which $\mathcal{A}\omega_i = \omega_i\mathcal{A}$, for $i = 0, 1, 2$. Hence, for each index i, there exists an automorphism ρ_i of $\mathcal{A}$ such that $\omega_i a = \rho_i(a)\omega_i$, for all $a \in \mathcal{A}$.

Since $\mathrm{SU}_q(2)$ is a compact quantum group, it admits a Haar integral h. Also, there is an automorphism ρ of $\mathcal{A}$ such that $h(ba) = h(\rho(a)b)$, for all $a, b \in \mathcal{A}$. We define a one-dimensional, left-invariant linear functional $\int$ on Ω by setting $\int a_0\omega_0 + a_1\omega_1 + a_2\omega_2 = h(a_1) + h(a_2)$. This functional is closed. To see this, observe first that there exist linear functionals χ_0, χ_1, χ_2 on $\mathcal{A}$ such that $\mathrm{d}a = \sum_{i=0}^{2}(\chi_i * a)\omega_i$, for all $a \in \mathcal{A}$. Since $d1 = 0$, we have $\chi_i(1) = 0$, for all i. Using this, and right-invariance of h, we get $\int \mathrm{d}a = h(\chi_1 * a) + h(\chi_2 * a) = h(a)\chi_1(1) + h(a)\chi_2(1) = 0$.

We claim now that $\int$ is not a twisted graded trace. Otherwise, let σ denote its twist automorphism. Then $\int \omega a = \int \sigma(a)\omega$, for all $\omega \in \Omega_1$. Therefore, for $a, b \in \mathcal{A}$ and $i = 1, 2$, we have $h(\rho\rho_i(a)b) = h(b\rho_i(a)) = \int b\rho_i(a)\omega_i = \int b\omega_i a = \int \sigma(a)b\omega_i = h(\sigma(a)b)$. Faithfulness of h now implies that $\rho\rho_i(a) = \sigma(a)$, for all $a \in \mathcal{A}$. Hence, $\rho_1 = \rho_2$. But if α, γ are the canonical generators of $\mathrm{SU}_q(2)$ as in [8], then $\rho_1(\alpha) = q^{-2}\alpha$ and $\rho_2(\alpha) = q^{-1}\alpha$, by Table 1 of [8]. Hence, $\rho_1 \neq \rho_2$. This contradiction shows that, as claimed, $\int$ is not a twisted graded trace.

We now truncate Woronowicz's calculus to get a one-dimensional differential calculus (Ω', d') over $\mathcal{A}$. Then (Ω', d') is a non-degenerate, left-covariant, one-dimensional calculus over $\mathcal{A}$, and $\omega_0, \omega_1, \omega_2$ is a linear basis for the space of invariant 1-forms.

The restriction $\int'$ of $\int$ to Ω' is a closed, left-invariant, one-dimensional linear functional on Ω'. As we saw is the case for $\int$, the functional $\int'$ is also not a twisted graded trace. This shows that the faithfulness hypothesis in Theorem 5.10 is necessary.

Lemma 5.11. *Let $\int$ be a left-invariant twisted graded trace on the universal unital differential calculus $(\tilde{\Omega}, d)$ over a Hopf algebra $\mathcal{A}$ admitting a Haar integral h. Let I be the left kernel of $\int$ and let $J = I \cap \tilde{\Omega}^{\text{inv}}$. Then the linear map from $\mathcal{A} \otimes J$ to I that sends $a \otimes \omega$ onto $a\omega$ is an isomorphism of left $\mathcal{A}$-modules. Hence, I is invariant under $\Delta_{\tilde{\Omega}}$ in the sense that $\Delta_{\tilde{\Omega}}(I) \subseteq \mathcal{A} \otimes I$.*

Proof. Let $\omega \in I$; using the identification of $\mathcal{A} \otimes \tilde{\Omega}^{\text{inv}}$ with $\tilde{\Omega}$, we write, as we may, $\omega = \sum_{i=1}^{M} a_i\omega_i$, where $a_1, \dots, a_M$ are linearly independent elements of $\mathcal{A}$, and $\omega_1, \dots, \omega_M$ are left-invariant elements of $\tilde{\Omega}$. Set $X = \{(h(ba_1), \dots, h(ba_M))|b \in \mathcal{A}\}$. We claim that $X = \mathbf{C}^M$. Suppose otherwise (and we shall obtain a contradiction). Then there exists a non-zero linear functional τ on $\mathbf{C}^M$ such that $\tau(x) = 0$, for all $x \in X$. Clearly, τ is determined by scalars $\mu_1, \dots, \mu_M$, in the sense that $\tau(\lambda_1, \dots, \lambda_M) = \sum_{i=1}^{M} \lambda_i\mu_i$, for all $\lambda_1, \dots, \lambda_M \in \mathbf{C}$. Moreover, since $\tau \neq 0$, the scalars μ_i are not all equal to zero. Now let $b \in \mathcal{A}$. Then $h(b(\sum_{i=1}^{M} \mu_i a_i)) = \sum_{i=1}^{M} \mu_i h(ba_i) = \tau(h(ba_1), \dots, h(ba_M)) = 0$. Hence, $\sum_{i=1}^{M} \mu_i a_i = 0$, by faithfulness of h. This contradicts the linear independence of the elements $a_1, \dots, a_M$. Consequently, to avoid contradiction, we must have $X = \mathbf{C}^M$. It follows that there exist elements $b_1, \dots, b_M \in \mathcal{A}$ such that $h(b_j a_i) = \delta_{ji}$, for $i, j = 1, \dots, M$. Hence, for any invariant element η in $\tilde{\Omega}$, we have, since $\omega \in I$, $0 = \sum_{i=1}^{M} \int \eta b_j a_i \omega_i = \sum_{i=1}^{M} h(b_j a_i) \int \eta\omega_i = \int \eta\omega_j$. Therefore, for any element $a \in \mathcal{A}$, $\int a\eta\omega_j = h(a) \int \eta\omega_j = 0$. Consequently, the form ω_j belongs to I and therefore, since it is left-invariant, it belongs to J. The lemma now follows. □

Theorem 5.12. *Let $\int'$ be an N-dimensional, left-invariant, closed twisted graded trace on the universal unital differential calculus $(\tilde{\Omega}, d)$ over a Hopf algebra $\mathcal{A}$ admitting a Haar integral h. The N-dimensional differential calculus (Ω, d) associated to $(\tilde{\Omega}, d, \int')$ is left-covariant and the canonical twisted graded trace $\int$ on (Ω, d) is left-invariant.*

Proof. Let ϕ be the canonical algebra isomorphism from $A \otimes \Omega$ onto the quotient algebra $(\mathcal{A} \otimes \tilde{\Omega})/(\mathcal{A} \otimes I)$ obtained by mapping $a \otimes (\omega + I)$ onto $a \otimes \omega + \mathcal{A} \otimes I$, for all $a \in \mathcal{A}$ and $\omega \in \tilde{\Omega}$. Then the map, $\Delta_\Omega : \Omega \to \mathcal{A} \otimes \Omega$, defined by setting $\Delta_\Omega(\omega + I) = \phi^{-1}(\Delta_{\tilde{\Omega}}\omega + \mathcal{A} \otimes I)$, for all $\omega \in \tilde{\Omega}$, is a coaction making (Ω, d) left-covariant. This follows from the readily verified facts that Δ_Ω is an algebra homomorphism extending the co-multiplication on $\mathcal{A}$ and that $(\mathrm{id}_{\mathcal{A}} \otimes d)\Delta_\Omega = \Delta_\Omega d$.

To see that $\int$ is left-invariant, let $\omega \in \tilde{\Omega}$ and suppose that $\Delta_{\tilde{\Omega}}(\omega) = \sum_{i=1}^{M} a_i \otimes \omega_i$, for some elements a_i in $\mathcal{A}$ and forms ω_i in $\bar{\Omega}$. Then

$$\begin{aligned}\left(\mathrm{id}_{\mathcal{A}} \otimes \int\right) \Delta_\Omega(\omega + I) &= \left(\mathrm{id}_{\mathcal{A}} \otimes \int\right)\left(\sum_{i=1}^{M} a_i \otimes (\omega_i + I)\right) \\ &= \sum_{i=1}^{M}\left(\int \omega_i + I\right) a_i = \sum_{i=1}^{M}\left(\int' \omega_I\right) a_i \\ &= \left(\mathrm{id}_{\mathcal{A}} \otimes \int'\right)(\Delta_{\tilde{\Omega}}\omega) = \left(\int' \omega\right) 1 = \left(\int \omega + I\right) 1.\end{aligned}$$

Thus, $\int$ is left-invariant, as required. □

Let $\mathcal{A}$ be a Hopf algebra and $\bar{\mathcal{A}}$ the quotient linear space $\mathcal{A}/\mathbf{C}1$ with corresponding quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathbf{C}1$. For $a \in \mathcal{A}$, set $\bar{a} = \pi(a) \in \bar{\mathcal{A}}$. Define the left coaction $\bar{\Delta}$ of $\mathcal{A}$ on $\bar{\mathcal{A}}$ by setting $\bar{\Delta}(\pi(a)) = (\mathrm{id}_{\mathcal{A}} \otimes \pi)\Delta(a)$ for all $a \in \mathcal{A}$. Define the left coaction Δ_N of $\mathcal{A}$ on $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes N}$ as the tensor product left coaction $\Delta \otimes \bar{\Delta}^{\otimes N}$ (see [4, 1.3.2 Eq. (61)] for the tensor product of two right coactions and adapt it to left coactions in the obvious way).

If $\varphi : \mathcal{A}^{N+1} \to \mathbf{C}$ is a multilinear function that vanishes on any element $(a_0, a_1, \dots, a_N)$, whenever any of the components $a_1, \dots, a_N$ belongs to $\mathbf{C}1$, we let $\hat{\varphi}$ be the corresponding linear map on $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes N}$ (so that $\hat{\varphi}(a_0 \otimes \bar{a}_1 \cdots \otimes \bar{a}_N) = \varphi(a_0, \dots, a_N)$). We say that φ is *left-invariant* if $(\mathrm{id}_{\mathcal{A}} \otimes \hat{\varphi})\Delta_N(c) = \hat{\varphi}(c)1$, for all $c \in \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes N}$, where 1 is the unit of $\mathcal{A}$.

Suppose now that φ is the twisted cyclic cocycle associated an N-dimensional, closed twisted graded trace $\int$ on Ω, for some left-covariant differential calculus (Ω, d) over $\mathcal{A}$. A straightforward calculation shows that

$$\left(\mathrm{id} \otimes \int\right)(\Delta(a_0)\Delta_\Omega d(a_1) \cdots \Delta_\Omega d(a_N)) = (\mathrm{id} \otimes \hat{\varphi})(\Delta_N(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_N)),$$

for all elements $a_0, a_1, \dots, a_N \in \mathcal{A}$. From this it follows easily that $\int$ is left-invariant if, and only if, $\hat{\varphi}$ is left-invariant.

We summarize our observations here in the following result.

Theorem 5.13. *Suppose that (Ω, d) is a left-covariant differential calculus over a Hopf algebra $\mathcal{A}$ and that $\int$ is an N-dimensional closed, twisted graded trace on Ω. Let φ be the corresponding twisted cyclic N-cocycle. Then $\int$ is left-invariant if, and only if, φ is left-invariant.*

6. A construction of a three-dimensional differential calculus

In this section, we indicate how our construction of a differential calculus from a closed twisted graded trace on the universal unital differential calculus can be used to show the existence of a three-dimensional calculus first constructed by very different means by Woronowicz.

First, recall that the universal unital differential calculus $\tilde{\Omega}$ over a Hopf algebra $\mathcal{A}$ is left-covariant. Let κ be the co-inverse on $\mathcal{A}$, and denote by m the linear map from $\mathcal{A} \otimes \tilde{\Omega}$ to $\tilde{\Omega}$ that sends the elementary tensor $a \otimes \omega$ onto the product $a\omega$. Define the linear map w from $\mathcal{A}$ to $\tilde{\Omega}_1^{\mathrm{inv}}$ by setting $w(a) = m(\kappa \otimes d)\Delta(a)$. If the unit 1 of $\mathcal{A}$ and the family

$(e_i)_{i\in I}$ form a linear basis for $\mathcal{A}$, then, for each positive integer k, the products of the form $w(e_{i_1})\cdots w(e_{i_k})$, where $i_1,\ldots,i_k \in I$, form a linear basis of $\tilde{\Omega}_k^{\mathrm{inv}}$ [10, Section 5 and 4, Section 14.3.2].

If $\mathcal{A}$ is a Hopf $*$-algebra, then $\tilde{\Omega}$ is a $*$-differential calculus over $\mathcal{A}$, where $w(a)^* = -w(\kappa(a)^*)$, for all $a\in\mathcal{A}$.

Suppose now that q is a non-zero real parameter for which $|q|\le 1$. We denote by $\mathcal{A}_q$ the Hopf algebra associated to the compact quantum group $\mathrm{SU}_q(2)$ [8]. Recall that $\mathcal{A}_q$ is the universal unital $*$-algebra generated by a pair of elements α and γ satisfying the relations

$$\alpha^*\alpha+\gamma^*\gamma=1,\quad \alpha\alpha^*+q^2\gamma\gamma^*=1,$$
$$\gamma^*\gamma=\gamma\gamma^*,\quad \alpha\gamma=q\gamma\alpha,\quad \alpha\gamma^*=q\gamma^*\alpha.$$

The co-multiplication Δ on $\mathcal{A}_q$ is the unique unital $*$-homomorphism for which $\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma$ and $\Delta(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma$.

Let $\mathbf{E} = (\mathbf{Z}\times\mathbf{N}\times\mathbf{N})\setminus\{(0,\ 0,\ 0)\}$. For $\varepsilon=(k,l,m)\in\mathbf{E}$, denote by a_ε the product $\alpha^k\gamma^l(\gamma^*)^m$. Here we use the usual convention in this context that for $k<0$, $\alpha^k=(\alpha^*)^{-k}$. It is well-known that these elements a_ε, together with 1, form a linear basis for $\mathcal{A}_q$. Writing w_ε for $w(a_\varepsilon)$, it follows that the products $w_{\varepsilon_1}w_{\varepsilon_2}w_{\varepsilon_3}$ form a basis for $\tilde{\Omega}_3^{\mathrm{inv}}$, that we shall call the *standard* basis of $\tilde{\Omega}_3^{\mathrm{inv}}$.

Again suppose that $\varepsilon=(k,l,m)$. We set $c(\varepsilon)=0$ if l or m are positive and we set $c(\varepsilon)=c(k)=(1-q^{-2k})(1-q^{-2})^{-1}$, if $l=m=0$. If ω is a standard basis element, $\omega=w_{\varepsilon_1}w_{\varepsilon_2}w_{\varepsilon_3}$, we set $c(\omega)=c(\varepsilon_1)+c(\varepsilon_2)+c(\varepsilon_3)$.

We shall say that ε is *reduced* if $(k,l)=(0,1)$, $(0,0)$ or $(1,0)$; in this case we set $t(\varepsilon)=-1$, 0, or 1, respectively, and we call $t(\varepsilon)$ the *type* of ε.

We shall say that a standard basis element $\omega=w_{\varepsilon_1}w_{\varepsilon_2}w_{\varepsilon_3}$ is *reduced*, if all the factors have reduced indices and their types are distinct. We set $t(\omega)=(t(\varepsilon_1),t(\varepsilon_2),t(\varepsilon_3))$.

Using Theorem 5.3, we define a three-dimensional left-invariant linear functional $\int$ on the universal unital differential calculus $\tilde{\Omega}$ over $\mathcal{A}_q$ by setting $\int$ equal to zero on all of the non-reduced standard basis elements, and by defining $\int$ on a reduced standard basis element $\omega=w_{\varepsilon_1}w_{\varepsilon_2}w_{\varepsilon_3}$ as follows:

(1) if $t(\omega)=(-1,\ 0,\ 1)$, $\int\omega=c(\omega)$;
(2) if $t(\omega)=(-1,\ 1,\ 0)$, $\int\omega=-q^4c(\omega)$;
(3) if $t(\omega)=(0,\ -1,\ 1)$, $\int\omega=-q^4c(\omega)$;
(4) if $t(\omega)=(0,\ 1,\ -1)$, $\int\omega=q^6c(\omega)$;
(5) if $t(\omega)=(1,\ -1,\ 0)$, $\int\omega=q^6c(\omega)$;
(6) if $t(\omega)=(1,0,-1)$, $\int\omega=-q^{10}c(\omega)$.

Using the formula $w(a)^*=-w(\kappa(a)^*)$, it is not that hard to prove that the functional $\int$ is self-adjoint.

Let σ_0 be the twist automorphism associated to the Haar measure h on $\mathcal{A}_q$; that is, σ_0 is the unique automorphism on $\mathcal{A}_q$ for which $h(a'a)=h(\sigma_0(a)a')$, for all $a,a'\in\mathcal{A}_q$. Let σ_1 be the unique automorphism on $\mathcal{A}_q$ for which $\sigma_1(\alpha)=q^{-4}\alpha$, $\sigma_1(\gamma)=q^{-4}\gamma$, $\sigma_1(\alpha^*)=q^4\alpha^*$ and $\sigma_1(\gamma^*)=q^4\gamma^*$. (This automorphism exists as a consequence of the universal property enjoyed by $\mathcal{A}_q$). Finally, set $\sigma=\sigma_0\sigma_1$; of course, σ is again an automorphism. Using [4, 14.3.2 Eq. (51)], one checks that $\int\omega a=\int\sigma(a)\omega$, for all $a\in\mathcal{A}_q$ and $\omega\in\tilde{\Omega}_3$.

If $a \in \mathcal{A}_q$ and $\Delta(a) = \sum_i b_i \otimes c_i$, then $\mathrm{d}(w(a)) = \sum_i w(b_i) w(c_i)$ [4, 14.3.2 Eq. (52)]. After a tedious case by case verification, this formula allows us to prove that $\int$ is closed.

Fully detailed proofs of these facts can be found in [5].

Now one uses Theorem 2.1 to deduce that $\int$ is a twisted graded trace. Moreover, the twist automorphism $\tilde{\sigma}$ of $\int$ extends the automorphism σ of $\mathcal{A}_q$. We use these facts, and the fact that $\int$ is self-adjoint, to apply the construction of Section 2 to the triple $(\tilde{\Omega}, d, \int)$ to deduce the existence of a left-covariant, three-dimensional $*$-differential calculus Ω over $\mathcal{A}_q$. We shall denote the canonical twisted graded trace on Ω by the same symbol $\int$ and refer to the domains of these functionals to distinguish them in cases of ambiguity.

Let π denote the quotient map from $\tilde{\Omega}$ onto Ω. It is easy to verify from the definition of $\int$ on $\tilde{\Omega}$ that,

(1) For all $k \in \mathbf{Z}$, $\pi(w(\alpha^k)) = c(k)\pi(w(\alpha))$, $\pi(w(\alpha^k \gamma)) = \pi(w(\gamma))$ and $\pi(w(\alpha^k \gamma^*)) = \pi(w(\gamma^*))$;
(2) For all $k, l, m \in \mathbf{Z}$ for which $l, m \geq 0$ and $l + m \geq 2$, we have $\pi(w_{(k,l,m)}) = 0$.

Set $\omega_0 = -q\pi(w(\gamma^*))$, $\omega_1 = \pi(w(\alpha))$ and $\omega_2 = -q^{-1}\pi(w(\gamma))$. It follows from Conditions (1) and (2) that ω_0, ω_1 and ω_2 linearly span Ω^{inv}. It is immediate from the definition of $\int$ on $\tilde{\Omega}$ that

$$\int \omega_0\omega_1\omega_2 = 1, \qquad \int \omega_0\omega_2\omega_1 = -q^4, \qquad \int \omega_1\omega_0\omega_2 = -q^4,$$
$$\int \omega_1\omega_2\omega_0 = q^6, \qquad \int \omega_2\omega_0\omega_1 = q^6, \qquad \int \omega_2\omega_1\omega_0 = -q^{10} \tag{6.1}$$

and that $\int \omega_i\omega_j\omega_k = 0$, for every $i, j, k \in \{0, 1, 2\}$, where any two of the indices i, j, k are the same.

Since the trace $\int$ on Ω is left faithful, it follows easily that ω_0, ω_1 and ω_2 are linearly independent and therefore that they form a linear basis for Ω^{inv}.

Let a and $b_1, \ldots, b_M$ and $c_1, \ldots, c_M$ be elements in $\mathcal{A}_q$ such that $\Delta(a) = \sum_{i=1}^M b_i \otimes c_i$. Then, by Eqs. (51) and (52) of [4, 14.3.2], and the equation $w(a)^* = -w(\kappa(a)^*)$, which holds for all $a \in \mathcal{A}_q$, we have

(1) $\pi(w(a))b = \sum_{i=1}^M b_i \pi(w(\bar{a}c_i))$, for all $b \in \mathcal{A}_q$;
(2) $\mathrm{d}a = \sum_{i=1}^M b_i \pi(w(c_i))$;
(3) $d\pi(w(a)) = \sum_{i=1}^M \pi(w(b_i))\pi(w(c_i))$;
(4) $\omega_0^* = q\omega_2,\ \omega_1^* = -\omega_1,\ \omega_2^* = q^{-1}\omega_0$.

Applying these formulas in our particular case, it is easy to check that the differential calculus (Ω, d) that we have constructed here satisfies the formulas in Tables 1, 2 and 6 of [8]. Using faithfulness of $\int$ on Ω, combined with the formulas in Eq. (6.1), one can readily verify that our differential calculus also satisfies the formulas of Table 5 of [8] and that the three elements $\omega_0\omega_1$, $\omega_0\omega_2$ and $\omega_1\omega_2$ form a linear basis for Ω_2^{inv}.

With this information at hand, it is now straightforward to conclude that our $*$-differential calculus (Ω, d) is isomorphic to the three-dimensional calculus constructed by Woronowicz in [8] by an entirely different method.

We believe that our method for constructing calculi is one that is perhaps more natural than other methods, since the basis of our approach is essentially to devise a “presentation” of the calculus in terms of generators and relations. It has the advantage over other methods that after some tedious but basic combinatorical computations, the structure of the whole space of differential forms is set up correctly.

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